Ex-post funding: How should a resource-constrained non-profit organization allocate its funds?

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We study the funds allocation problem for a resource-constrained non-profit organization (NPO) that implements social development projects for public good. In addition to raising funds from donors who contribute prior to project implementation (“traditional donors”), the NPO uses a novel approach, which we term as the “ex-post funding” approach, to also raise funds from donors who contribute based on the results delivered by the NPO (“ex-post donors”). In this approach, the NPO uses its initial funds to implement early phases of the project, creates “results-certificates” from the completed phases, and invites ex-post donors to purchase these certificates. The donations raised from selling the results-certificates are used to recover the NPO’s own funds used in the project implementation. Operationalizing this approach is complicated when the project must incur a large fixed cost before any benefits are delivered by the project and the total benefit delivered is time sensitive. We show that for a given amount of initial funds available, there exists a threshold amount of funds that the NPO should raise from traditional donors before implementing the project phases so as to maximize the total expected benefit delivered. Through numerical studies, we analyze how the threshold of funds raised from traditional donors and the total benefit delivered vary with donor characteristics such as donor willingness to give and the proportion of donors who contribute prior to project implementation. Our numerical studies suggest that even with relatively small amount of initial funds, the NPO can deliver substantially higher benefit by using the ex-post funding approach when compared to using a traditional approach that requires the NPO to raise all the funds required upfront.

Key words: non-profit operations, resource allocation, donor funding

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1. Introduction

Most non-profit organizations (NPOs) raise funds for operations through donations from various entities and use these funds to achieve social goals, where the social goals typically involve the provision of a public good that simultaneously benefits many individuals. Donors who contribute to the NPO’s cause include government agencies, philanthropic institutions, and individual donors. In this paper, we focus our attention on the operations of an early-stage, resource-constrained NPO\textsuperscript{1} that primarily relies on individual donor contributions to fund its operations.

While NPOs are reliant on donor contributions, donors are usually unsure about donating to the NPO’s cause due to lack of adequate information on how effective and efficient the NPO is [The William and Flora Hewlett Foundation and McKinsey & Company, 2008]. A web-based survey conducted by a major US charity, United Way of America, found that a large proportion of donors who lacked trust in NPOs cited “lack of transparency” as the key concern [United Way of America, 2005]. As a result, the availability of donor contributions is highly uncertain and NPOs have to manage the impact of this uncertainty on project implementation. Typically, NPOs wait till all the funds required for the project are raised, before starting the project implementation.

The emergence of intermediaries, such as Charity Navigator\textsuperscript{2} and Give India\textsuperscript{3} has helped address the issue of lack of transparency to some extent and help connect NPOs with donors. These intermediaries then provide information to donors by rating NPOs in terms of their use of funds, accountability, transparency, and financial sustainability. While the NPO rating platforms provide some transparency, they do not provide detailed information about the NPO’s ability to deliver results that donors care about. To overcome this, some NPOs are adopting a relatively new and emerging approach, which we term as the “ex-post funding” approach. In this approach, the NPOs use their initial funds, even if it is insufficient to meet the total cost of the project, to implement early phases of the project. The outcomes from the implemented phases, along with the costs

\textsuperscript{1} Henceforth, by NPO we mean an early-stage, resource-constrained NPO.

\textsuperscript{2} http://www.charitynavigator.org

\textsuperscript{3} http://www.giveindia.org
incurred, are documented and made available for donors to evaluate the NPO’s ability to deliver results and make their contribution decisions. For example, United Care Development Services (UCDS)\(^4\), a NPO based in India, creates detailed “results-certificates” that document the results and costs incurred for finished phases of a project. The NPO puts these results-certificates on its website and invites donors to ‘purchase’ these certificates [Mehrotra et al., 2014]. Other examples include Splash\(^5\), a NPO that aims to provide clean drinking water for children and maintains a monitoring and evaluation platform called “Proving It”\(^6\) where donors can track the progress of each project before making their donations, and Educate Girls, a Dasra\(^7\) portfolio organization, which is using a pay-by-results approach wherein donors release payments only upon achieving clearly identified milestones.

The ex-post approach makes the NPO’s operations more transparent to the donors and allows the donors to evaluate the quality of the NPO as well as the output of the given project. The funds raised against results already achieved are used to recover the original funds and implement future phases of the project. Implementing the project in phases and raising funds against finished phases also allows the NPO to raise donations in smaller amounts, through multiple rounds of fund raising. As Vesterlund [2006] suggests, raising small contributions over multiple rounds typically leads to a higher amount of overall funds raised. The ex-post funding approach, however, exposes the NPO to some risks. Because the NPO uses its initial funds to implement only a portion of the project, completion of the project is dependent on being able to raise contributions against results delivered. Also, donors are heterogeneous in their giving behavior with some donors more willing to contribute to projects that are yet to be implemented while others may prefer to donate to a NPO only after observing the NPO’s performance. Thus, the NPO’s ability to raise contributions during different stages of the project, before, during and after, completion of the various phases depends on the mix of donors in the population.

\(^4\) http://www.yousee.in
\(^5\) http://www.splash.org
\(^6\) http://www.proving.it
\(^7\) http://www.dasra.org/
The use of an ex-post funding approach is further complicated when the projects involve significant upfront fixed costs. As discussed by Andreoni [1998], Bagnoli and McKee [1991] and Vesterlund [2006], many public goods may be of the threshold type where a minimum fixed cost, in addition to operating costs during implementation, needs to be incurred before any implementation can begin and output delivered. In such a situation, the NPO has to raise funds to cover the fixed costs before it can start implementation and show results. The NPO can choose to delay the implementation of the project phases by raising funds in excess of the fixed costs early on as these additional funds act as a buffer against uncertain availability of funds during the implementation stage. The traditional approach of NPOs raising all the funds needed to implement the project fully is an example where the NPO completely buffers against uncertainty. However, raising more donations upfront before it starts implementation will delay the delivery of results. When the benefits delivered are time sensitive, the NPO faces a trade-off when making its decision on how much buffer to accumulate before starting the implementation. As the initial funds available with the NPO also help reduce the NPO’s dependence on flow of donor contributions to continue implementation of future phases of the project, the NPO needs to account for this when deciding on how much buffer to have.

The interactions between the donors, NPOs, and beneficiaries in the NPO-value-chain are varied and the challenges faced by NPOs in their operations are numerous. This paper focuses on one particular operational issue, namely the use of initial funds, that can help improve the effectiveness of NPO operations and adds to the growing literature in the area of non-profit operations management. The research questions in this paper are motivated by the trade-off faced by an NPO between delaying the start of the implementation phase and reducing dependence on the flow of donor contributions during the implementation phase. The time sensitivity of benefits, the relative willingness of donors to contribute towards ongoing projects versus their willingness to contribute for projects that are yet to be implemented and the NPO’s initial funds affect this trade-off. Specifically, we address the following questions: (i) How should the NPO decide on the buffer of funds to

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For example, raising funds for constructing a new school and bearing the subsequent monthly expenses of operating the school until the number of enrolled students reaches a certain number that ensures self-sustenance [Andreoni, 1998]. In this example, the construction of school is the fixed phase and the monthly expense of running the school for the required period is the implementation phase.
raise before starting the implementation stage of the project, and (ii) How is this decision affected by project, donor and NPO characteristics?

We assume the NPO’s utility is aligned with maximizing the total benefit provided (output from the project) and use the total benefit delivered as the basis of comparison. In our setting, a project is characterized by its fixed cost, number of phases in the implementation stage, per phase implementation cost and benefit, and the time sensitivity of benefits. The NPO is characterized by the initial funds available to implement the ex-post funding approach. We model two types of donors, ‘traditional’ and ‘ex-post’ donors. Traditional donors contribute to the NPO based on their beliefs about the NPO rather than actual benefits delivered by the NPO from implemented phases while the ex-post donors only contribute against results delivered. As stated earlier, our focus in this paper is on understanding the operational decisions that the NPO needs to make when using an ex-post funding approach. Consequently, we abstract from donor behaviour details and model the donation process using aggregate probabilities of raising funds during the different stages of the project from the two kinds of donors and the proportion of traditional donors. Using this framework, we find that there exists an unique optimal level, possibly zero, upto which the NPO should accumulate funds from traditional donors before starting the implementation stage. Through numerical studies, we also explore the effectiveness of the ex-post funding approach when compared to the traditional approach where the NPO raises all the funds required to complete the project upfront. We find that even when the NPO has limited initial funds, using an ex-post funding approach allows the NPO to deliver significantly higher expected benefit compared to the traditional approach.

The rest of the paper is organized as follows. In §2, we position our work in relation to the extant literature. We then discuss our modeling framework and the analysis of the ex-post funding approach in §3. Results of our numerical studies to understand how various parameters affect the optimal allocation decision and performance of the ex-post funding approach are presented in §4. We discuss the implications of our results and future research directions in §5. Appendix A contains the proofs for all our results.
2. Literature Review

Our paper broadly touches four streams of literature. The first stream is related to project management focused on stochastic resource-constrained project scheduling. There is vast literature in the project management stream that focuses on the impact of resource constraints and uncertainty on the project completion time [Brucker et al., 1999, Kolisch et al., 1995]. Herroelen and Leus [2005] review the fundamental approaches for scheduling under uncertainty: reactive, robust (proactive), fuzzy and stochastic. Terwiesch and Loch [1999] build on the analytical work of Krishnan et al. [1997] to study the effect on project completion time of overlapping project activities in the global electronics industry. We build on this existing literature to study the impact of uncertainty in availability of funds on the project completion time. As our paper focuses on the social development projects, the project completion time affects the total benefit delivered by a NPO. Also, unlike uncertainty models studied in existing literature [for example, Terwiesch et al., 2002], the uncertainty in fundraising faced by a NPO is exogenous in nature (due to its dependence on donors). By using an ex-post approach and having the optimal amount of buffer funds, the NPO effectively decouples the uncertainty in resource availability from the project schedule. The NPO can use the reserved resources for implementing those phases for which the funds raised through donations during the implementation stage may be insufficient.

The second stream is related to social enterprises, which largely comprises of literature classifying the various non-profit models seen in practice and discusses the opportunities and challenges associated with these enterprises [for example, Uvin et al., 2000, Borzaga and Defourny, 2001, Sriram, 2011, Peredo and McLean, 2006]. Canavan et al. [2008] provide a review of the literature on performance based financing, a form of ex-post funding approach. In the context of addressing the Millenium Development Goals (MDG), Oxman and Fretheim [2009] argue that payment for results programs must be designed carefully and targeted at the right levels for them to have an impact while Kusek and Rist [2004] provide a framework for development practitioners to undertake results based monitoring and evaluation. Our paper focuses on a specific challenge faced by NPOs, namely that of maximizing social benefit, when using the ex-post funding approach and
uses a mathematical model to derive insights regarding how to manage the operational trade-offs. Further, we also do not consider situations where a NPO can use for-profit activities to generate surplus and use the surplus to fund social or mission activities. As such, the NPO’s operations in our context are completely funded by donor contributions. There is a vast literature related to donor funding and how charities can design fund raising strategies to maximize contributions from donors. Vesterlund [2006] provides an excellent review of the literature dealing with donor giving behavior and discusses theoretical, empirical and experimental studies that have been used to understand donor motivations for giving. Our focus in the current paper is on operational decisions that a NPO needs to make while implementing development projects and using a specific fund raising strategy, and not so much on designing the fund raising strategy. As a result, we model the arrival of funds to the NPO as an exogenous process. Our modeling assumptions about the donation process are consistent with observed empirical and experimental evidence. Empirical evidence from prior research on whether efficiency is rewarded in the donor marketplace is equivocal [see van Iwaarden et al., 2009, Kaplan, 2001, Parsons, 2003, for instance], pointing to heterogeneity among donors about the impact of NPO efficiency and effectiveness on their contribution decisions. By modeling different kinds of donors, we attempt to capture the heterogeneity in the donor marketplace and analyze its effect on the NPO’s operational decisions.

The third stream is related to the emerging non-profit, humanitarian-operations literature in Operations Management. A comprehensive survey of the literature related to non-profit operations management can be found in Feng and Shanthikumar [2016]. In this stream De Véricourt and Lobo [2009] look at resource management for a non-profit firm that engages in both for-profit and mission (non-profit) activities, where revenues generated from the for-profit activities are used to subsidize the mission activities. When the ultimate objective for the firm is maximizing social impact, they model the trade-off between achieving social impact in the current period versus generating additional financial resources for social impact in the future. They find that optimal resource allocation has a threshold policy wherein the firm should undertake mission activities only when the available financial resources are above a threshold. In contrast to De Véricourt
and Lobo [2009], we consider an NPO that undertakes only mission (non-profit) activities and relies on external donations for pursuing these activities, and the trade-off is between delaying the start of implementation and making the implementation stage less dependent on the external, uncertain flow of donations. In that sense, our paper is closer to Natarajan and Swaminathan [2012], who consider the impact of funding patterns on the management of a humanitarian supply-chain distributing nutritional products. They use a multi-period stochastic inventory model with financial constraints to study the effect of amount, schedule, and uncertainty of funding on the performance of the system. They find that a state-independent base stock policy is optimal. Their numerical experiments show that receiving early funding is better and under-funded systems that receive funds in a timely manner might be better than systems with delayed full funding. Our focus in the current paper is also on understanding how uncertainty in the availability of funds affects the social impact from a development project. The ex-post funding approach used by the NPO in our model has similarities with NPO audit contracts studied by Privett and Erhun [2011]. Privett and Erhun take the approach that funding methods are essentially contracts where the donor is exchanging money for social benefits, with the NPO organization acting as a seller of the social benefit. They argue that the model wherein donors allocate funds after scrutinizing financial statements and public reporting by NPOs is inefficient because these reports are unreliable. They propose and develop audit contracts that allow the donor an option to audit the non-profit organization for efficiencies and impose a penalty if there is a discrepancy between reported and actual efficiencies. Our paper complements Privett and Erhun [2011] in the sense that, in the ex-post funding approach, the donors can, if they choose, verify the authenticity of the results-certificates put up for sale by the NPO, and this approach therefore is a form of audit contract between the NPO and donor.

Finally, the fourth stream of literature focuses on the interaction between finance and operations in the for-profit business sector [Buzacott and Zhang, 2004, Chao et al., 2008]. In these models the firm has limited cash resources and its ability to raise money depends on its inventory levels. Instead of explicitly considering a budget constraint, Buzacott and Zhang [2004] model available cash in each period as a function of the firm’s assets and liabilities that are updated periodically through
production. Chao et al. [2008] consider a dynamic, periodic-review, inventory control problem of a self-financing retailer with lost sales. They characterize the optimal policy and develop an algorithm to compute the inventory levels in a given period. Dada and Hu [2008] consider the problem of a cash-constrained newsvendor when demand is uncertain, i.e., a liquidity-constrained nascent firm. In their setting the bank (lender) is the Stackelberg leader, deciding on interest rate to charge, and the newsvendor is the follower deciding on its procurement. The ex-post funding approach studied in this paper has similarities with the self-financing approach in that the implementation of future phases depends on the funds raised from previously completed phases, and the total benefit delivered is a function of the initial funds available with the NPO. The uncertainty in our model arises from the stochastic flow of donations in the non-profit sector.

3. Model and Analysis

We consider an NPO that implements development projects with the objective of providing a public good to a beneficiary population. The NPO is resource constrained along multiple dimensions. The primary constraint we focus on is limited financial resources wherein the initial funds available with the NPO are less than the total cost of implementing a project. Consequently, the NPO needs to raise funds in the form of contributions from donors to successfully complete the project. Other non-financial resource constraints, such as NPO effort, availability of labor resources, and technical know-how, are a key input in delivering the project output. In the context of our model, we assume these constraints are such that the NPO can implement at most one project at a time.

We model the NPO’s output as a threshold public good [Bagnoli and McKee, 1991, Andreoni, 1998], which involves fixed and variable operating costs. For instance, a public television or radio station needs expensive equipment before it begins to broadcast. Similarly, a sports arena or a school needs to be of a certain minimum size before it can be useful to beneficiaries [Andreoni, 1998]. Marks and Croson [1999] discuss delivery of many lumpy public goods (parks, community libraries) using this approach. Specifically, the NPO needs to incur an initial fixed cost, $F \geq 0$, before the project can start delivering benefits to the target population. Beyond the fixed costs, the NPO also
incurs on-going operating costs to deliver benefit to the beneficiaries. We call these two stages of the project as the fixed cost and implementation stages, respectively. The implementation stage consists of multiple phases, with $M > 0$ denoting the total number of phases required to complete the project. We normalize the operating cost for implementing each phase to 1 unit. As discussed in §1, implementing the project in multiple phases allows the NPO to seek repeated contributions for small amounts, allowing the NPO to raise larger total contributions. Operationally, it allows the NPO to continue implementing the project even when partial funds are available. We follow Privett and Erhun [2011], and model the NPO output (benefit delivered in each phase) as a scalar measure. While this representation is of course a simplification, it is nevertheless useful to illustrate the key trade-offs faced by the NPO. Further, we also assume throughout the paper that the NPO’s objective is aligned with those of the beneficiaries of the project; i.e., the NPO maximizes the total output (or, benefit) delivered by the project. The output delivered by each completed phase is denoted by $u$, with $u > 0$. For ease of exposition, we assume $u$ is the same for each phase. There is a time value associated with the benefit, i.e., it is better for the beneficiaries if the project is implemented sooner than later, and we use a discount factor, denoted by $\delta$ with $0 < \delta \leq 1$, to model the time value of benefits delivered. Because the NPO must wait for the necessary funds to become available before it can continue with the implementation of the project, the financial resource constraints have an impact on the total number of periods required to complete the project and thereby the actual discounted benefit delivered by the project.

To model heterogeneity in donor behavior, we model the donor population as consisting of two types of donors that we term “traditional donors” and “ex-post donors”. Traditional donors are those donors who are willing to contribute in anticipation of the NPO delivering results and do not need the NPO to have delivered results; that is, these donors donate ex-ante in anticipation. In contrast, ex-post donors consist of those donors who donate only against actual benefits (or, results) delivered by the NPO. Let $\gamma \in (0,1]$ be the fraction of traditional donors in the donor population.
While traditional donors contribute in anticipation of results, it is possible that the donors’ willingness to contribute changes as the project progresses. For instance, some donors might be less willing to donate to a project that has already started delivering benefits and prefer to contribute to projects that are yet to start. To account for this, let $p_j(m)$ be the probability that $j$ units of funds are raised from the traditional donors in a given period, with $j = 0, 1, \ldots$, when $m$ phases of the project have been completed, for $0 \leq m < M$. Likewise, the willingness of ex-post donors to donate against results delivered in a particular phase might vary with the total benefit provided thus far. That is, the probability that an ex-post donor contributes for results from a completed phase, $\alpha_m$, varies with the total number of phases already completed. We assume ex-post donors contribute towards results from at most one phase in a given period (relaxing this assumption makes the analysis complicated without any additional insights) and $\alpha_0 = 0$.

### 3.1. Fixed cost stage

The NPO must first raise funds required to meet the fixed costs which are incurred even before the various phases of the project that actually deliver benefit are implemented. To raise funds to cover the fixed costs, the NPO undertakes fund raising drives in each period, till at least an amount sufficient to meet the fixed cost, $F$, is raised. Because $p_j(m)$ varies with $m$, the NPO can choose to raise funds strictly greater than what is required to meet the fixed costs before starting the implementation. Let $x_0 \geq 0$ denote the minimum amount of funds over and above the fixed cost $F$ that the NPO wants to raise before starting the implementation stage of the project (in addition to its own initial funds that are reserved for the implementation stage).

Let $\theta = F + x_0$ denote the minimum amount of funds the NPO wishes to raise before starting the implementation stage and $\tau_F(x_0)$ denote the duration of the fixed cost stage of a given project. During this stage, only traditional funding is available to the NPO and $\gamma \times p_j(0)$ is the probability that $j$ units of funds are raised in a given period, for $j = 0, 1, \ldots$. We have

$$\tau_F(x_0) = \min \left\{ \tau \text{ s.t. } \sum_{t=1}^{\tau} X_t \geq F + x_0 \right\}$$  \hspace{1cm} (1)$$

where $X_t$ is the funds raised in period $t$. 
Let \( X_F(x_0) = \sum_{i=1}^{\tau_F(x_0)} X_i - F \geq x_0 \) be the funds raised from traditional donors that are available for the implementation stage after incurring the fixed costs \( F \) required to start the project. Because \( X_F(x_0) \) is increasing in \( x_0 \), as \( x_0 \) increases the NPO has more funds available for the implementation phase and is less dependent on the funds raised during the course of the implementation stage. Thus, the time required to implement the \( M \) phases is stochastically decreasing in \( x_0 \). However, \( \tau_F(x_0) \) is stochastically increasing in \( x_0 \) and thus delays the beginning of the implementation stage and the NPO therefore needs to evaluate the trade-off in time required for the two stages of the project.

### 3.2. Implementation stage

During the implementation stage, the NPO can use its initial funds along with funds raised from both traditional and ex-post donors to fund the operating expenses for each phase. In any period during the implementation stage, let \( y \) denote the the NPO’s own funds (which are recovered through the donations from ex-post donors) that are currently available to implement future phases, \( x \) be the funds raised from traditional donors that are currently available to implement future phases, and \( m \) be the number of phases already completed.

In each period, the NPO implements the next phase of the project as long as it has funds available, irrespective of when these funds were received. We assume that the NPO uses any available funds raised from traditional donors to implement the next phase of the project. When the NPO doesn’t have any funds raised from traditional donors available, it uses its own funds that might be available to implement the project phase. If no funds are available at the beginning of a period, the NPO waits till a contribution is received before implementing the next phase. Contributions from ex-post donors are limited by the NPO’s own initial funds spent on implementation. That is, if a phase is implemented using contributions received from traditional donors, the NPO cannot use those results to raise funds from ex-post donors. Finally, the NPO implements all the \( M \) phases of the project and does not abandon a project after partial completion. This is a reasonable assumption.

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*From our discussions with NPOs, they do not want to be seen as indulging in ‘double-dipping’ where they seek contributions from ex-post donors for phases completed using funds contributed by traditional donors.*
if the NPO is interested in implementing projects that have similar characteristics in terms of costs and benefits delivered and diverting donor contributions to a different project does not increase the overall social benefit delivered.

Let $U(x, y, m)$ denote the total expected discounted utility delivered from the current project when the state is $(x, y, m)$. If $u$ is the utility delivered from each completed phase and $\delta$ is the discount factor, we have

$$U(x, y, m) = \begin{cases} 
  u + \gamma \left( \sum_{j=0}^{M-m-x} \delta U(x+j-1, y, m+1)p_j(m+1) \right) + \\
  (1-\gamma) \left( \delta U(x-1, y+1, m+1)\alpha(m+1) 
  + \delta U(x-1, y, m+1)(1-\alpha(m+1)) \right) \\
  if \ x > 0, \ y_0 > y \geq 0 and \ m < M, \\
  u + \gamma \left( \sum_{j=0}^{M-m-x} \delta U(x+j, y-1, m+1)p_j(m+1) \right) + \\
  (1-\gamma) \left( \delta U(x, y, m+1)\alpha(m+1) + \delta U(x, y-1, m+1)(1-\alpha(m+1)) \right) \\
  if \ x = 0, \ y_0 \geq y > 0 and \ m < M, \\
  \gamma \left( \sum_{j=0}^{M-m-x} \delta U(x+j, y, m)p_j(m) \right) + \\
  (1-\gamma) \left( \delta U(x, y+1, m)\alpha(m) + \delta U(x, y, m)(1-\alpha(m)) \right) \\
  if \ x = 0, \ y = 0 and \ m \leq M, \\
  u + \gamma \left( \sum_{j=0}^{M-m-x} \delta U(x+j-1, y, m+1)p_j(m+1) \right) + \\
  (1-\gamma) \left( \delta U(x-1, y, m+1) \right) \\
  if \ x > 0, \ y = y_0 and \ m < M, \\
  0 \\
  if \ m = M. 
\end{cases} \quad (2)$$

In the above expression for $U(x, y, m)$, we assume that in any given period contributions come from either the traditional donors or ex-post donors, but not both. Also, when $y = y_0$ it means either the NPO hasn’t used its initial funds to implement any of the completed phases yet (because it had funds raised from traditional donors available), or that it has received contributions from ex-post donors against all the phases implemented using its own initial funds. As a result, when $y = y_0$ at the end of a period, no contributions are received from ex-post donors.

Given the delay imposed because of having to raise funds to cover the fixed costs and accumulate funds from traditional donors before the start of the implementation stage, the total expected discounted utility delivered from the project for a given $x_0$ and $y_0$ is given by
\[ \bar{U}(x_0, y_0) \triangleq \mathbb{E} \left[ \delta^{\tau_F(x_0)} U(X_F(x_0), y_0, 0) \right], \quad (3) \]

where the expectation is over \( \tau_F(x_0) \) and \( X_F(x_0) \) jointly. Given our assumption that the NPO’s objective is aligned with that of the beneficiaries, the NPO would like to maximize \( \bar{U}(x_0, y_0) \).

The model as described above is analytically intractable even with the assumptions made. In order to derive insights about the trade-off that the NPO faces, we impose some additional restrictions on \( p_j(m) \). Specifically, we assume \( p_j(m) \) is as follows:

\[
p_j(m) = \begin{cases} 
0 & \text{for all } m, \text{ for all } j > 1, \\
0 & \text{for all } j > 0 \text{ for } m = M, \\
\beta_m & \text{for } m < M \text{ and } j = 1, \\
1 - \beta_m & \text{for } m < M \text{ and } j = 0.
\end{cases} \quad (4)
\]

While the above probability distribution appears restrictive, it is general enough because we can re-scale the time period (and the cost of implementing a phase, the benefit delivered and discount factor appropriately) such that the probability of receiving more than one unit of funds in a given period is negligible. In §4, we conduct extensive numerical studies to understand the trade-off in more general settings.

With the probability of contributions from traditional donors given by equation (4), we can see that \( X_F(x_0) = x_0 \) always. Further, \( \bar{U}(x_0, y_0) = \mathbb{E} \left[ \delta^{\tau_F(x_0)} U(x_0, y_0, 0) \right] \).

**Lemma 1.** When the probability distribution of contributions from traditional donors is given by equation (4),

\[
\bar{\delta}(x_0) \triangleq \mathbb{E} \left[ \delta^{\tau_F(x_0)} \right] = \left[ \frac{\delta \gamma / \beta_0}{1 - \delta \left[ \gamma (1 - \beta_0) + 1 - \gamma \right]} \right]^{(F + x_0)}. \quad (5)
\]

As equation (5) shows, \( \bar{\delta}(x_0) \) is decreasing in \( x_0 \); that is, as the NPO accumulates more funds before starting implementation, it pays a penalty because of larger discounting of the benefits delivered from implementation.

With the probability distribution given by equation (4), the expected utility \( U(x, y, m) \) given by equation (2) simplifies to

\[
\bar{U}(x_0, y_0) \triangleq \mathbb{E} \left[ \delta^{\tau_F(x_0)} U(X_F(x_0), y_0, 0) \right],
\]
\[
U(x, y, m) = \begin{cases}
\gamma \left( u + \delta [\beta_{m+1} U(x, y, m+1) + (1 - \beta_{m+1}) U(x-1, y, m+1)] +
(1 - \gamma) \left( u + \delta [\alpha_{m+1} U(x-1, y+1, m+1) + (1 - \alpha_{m+1}) U(x-1, y, m+1)] \right) \right)
+ (u + \delta U(x-1, y, m+1)) & \text{if } x > 0, y > 0, y_0 \geq y \geq 0 \text{ and } m < M, \\
\gamma \left( u + \delta [\beta_{m+1} U(x+1, y-1, m+1) + (1 - \beta_{m+1}) U(x, y-1, m+1)] +
(1 - \gamma) \left( u + \delta [\alpha_{m+1} U(x, y, m+1) + (1 - \alpha_{m+1}) U(x, y-1, m+1)] \right) \right)
+ (u + \delta U(x, y-1, m+1)) & \text{if } x = 0, y_0 \geq y > 0 \text{ and } m < M, \\
\gamma \left( u + \delta [\beta_{m+1} U(x, y, m+1) + (1 - \beta_{m+1}) U(x-1, y, m+1)] +
(1 - \gamma) \left( u + \delta U(x-1, y, m+1) \right) \right) & \text{if } x > 0, y = y_0 \text{ and } m < M, \\
0 & \text{if } m = M.
\end{cases}
\]  

For a given value of \(y_0\), continuing the implementation of project phases becomes less dependent on the flow of funds during implementation as \(x_0\) increases and allows the NPO to reduce the time required to implement the \(M\) phases. However, the delay in start of the implementation is increasing in \(x_0\) and \(\bar{\delta}(x_0)\) is decreasing in \(x_0\).

To determine the optimal \(x_0\) (for a given \(y_0\)), we need to understand the properties of \(\bar{\delta}(x_0)\) and \(U(x_0, y_0, 0)\). To this end, for a given \(y\) and \(m\), let \(\Delta_x(y, m) = U(x + 1, y, m) - U(x, y, m)\) for all \(x > 0\). As Theorem 1 shows, \(\Delta_x(y, m)\) is non-increasing in \(x\) for all \(y\) and \(m\).

**Theorem 1.** For any \(y\) and \(m\), the marginal benefit of a unit of funds from traditional donors is non-negative and is non-increasing in the amount of funds available. That is, \(0 \leq \Delta_{x+1}(y, m) \leq \Delta_x(y, m)\) for \(x > 0\), for all \(y\) and all \(m\).

Theorem 1 shows that there is diminishing marginal benefit from increasing the amount of funds raised from traditional donors before starting the implementation stage. Thus, for sufficiently large \(x_0\), the increase in \(U(x_0, y_0, 0)\) due to accumulating an additional unit of funds will not be adequate to compensate for the loss in overall utility because of a reduction in \(\bar{\delta}(x_0)\). Theorem 2 shows that \(\bar{U}(x_0, y_0)\) is unimodal in \(x_0\) and there exists a unique optimal value of \(x_0\) that maximizes the discounted expected utility.
Theorem 2. For a given $y_0$, there exists a unique $x_0^*(y_0)$, possibly 0, such that $x_0^*(y_0) = \arg\max_{x_0 \geq 0} \{ \bar{U}(x_0, y_0) \}$.

The funds raised from traditional donors and the NPO’s initial funds are both used to meet the operating expenses incurred when implementing the various phases of the project. Thus, $x_0$ and $y_0$ are substitutes and the marginal value of an additional unit of funds raised from traditional donors is non-increasing in the amount of initial funds available and vice versa. Theorem 3 formalizes this intuition and shows that $x_0^*$ is non-increasing in $y_0$.

Theorem 3. The surplus funds raised from traditional donors and the NPO’s initial funds available at the beginning of the implementation stage are substitutes. That is, $\Delta_x(y_0, 0)$ is non-increasing in $y_0$ and $x^*(y_0)$ is non-increasing in $y_0$.

4. Numerical Studies

The analytical results in §3 illustrate how the donor, project and NPO characteristics affect the trade-off faced by the NPO, albeit with specific assumptions regarding the donation process. In this section, we supplement the analytical results with an extensive numerical study in a more general setting to understand how various parameters affect (a) the optimal buffer of funds from traditional donors to accumulate before the start of the implementation stage, (b) the total expected utility delivered, and (c) the performance of the ex-post funding model compared to traditional approach of raising all the funds required for the project from traditional donors before starting the project.

The results of our numerical studies answering these questions are presented in §4.2–4.4, after describing the setup for our studies in §4.1. In §4.5, we conduct numerical studies to examine the decision of a resource constrained NPO to allocate its initial funds between the fixed cost stage and implementation stage in addition to the decision on the optimal amount of funds from traditional donors to accumulate at the beginning of the implementation stage.

4.1. Numerical study setup

Project characteristics. We consider development projects with multiple phases during implementation. For the results reported here, we set the fixed cost $F = 10$ and discount factor $\delta = 0.9$. 
While we do not report it, the results are qualitatively similar for other values of $F$ and $\delta$. We normalize the cost of implementing each phase to 1 unit, while the benefit delivered from each completed phase, $u$, is set to 100. From the previous analytical results, it is easy to see that the total expected benefit scales in proportion to $u$, and therefore the results are invariant to the choice of $u$.

**Donor characteristics.** The donor characteristics modeled in the numerical studies are summarized in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Fixed cost stage</th>
<th>Implementation stage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional donors</td>
<td>$\mathbb{P}(X_t = j) = \beta_0^j (1 - \beta_0)$</td>
<td>$\mathbb{P}(X_t = 1</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{P}(X_t = j</td>
<td>x, m) = \frac{\beta_m^j(1 - \beta_m)}{1 - \beta_m^{1-x+m}}$</td>
</tr>
<tr>
<td>Ex-post donors</td>
<td>$\mathbb{P}(X_t = 1</td>
<td>m) = \alpha_m$, $\mathbb{P}(X_t = 0</td>
</tr>
</tbody>
</table>

**Table 1** Summary of donor characteristics used in numerical studies.

We model the traditional donors through the probability of raising contributions. Specifically, we model the probability of raising $j$ units of funds in a given period during the fixed cost stage as $p_j(0) = \beta_0^j (1 - \beta_0)$. We vary $\beta_0$ in the numerical experiments to understand the impact of traditional donors’ willingness to contribute during the fixed cost stage. For the implementation stage, we consider two cases for the contributions received from traditional donors. In the first, in any period at most one unit of funds are raised from traditional donors and the probability of receiving funds is given by $\beta_m$ when $m$ phases have been completed. In the second, the probability of raising $j$ units of funds from traditional donors during the implementation stage when $m$ phases have been completed and the NPO has $x$ units of funds raised from traditional donors is given by $p_j(x, m) = \frac{\beta_m^j(1 - \beta_m)}{1 - \beta_m^{1-x+m}}$. In both the scenarios, we set $\beta_m = \beta_1 \times (1 - \eta_t(m/M))$ to capture the dependence of the probability of contributions on the number of phases completed. We vary $\beta_0$, $\beta_1$ and $\eta_t$ to quantify the impact of different traditional donor characteristics on performance.
We model the ex-post donors through the probability of selling results of a completed phase, $\alpha_m$, when $m$ phases have been completed. As in the analytical model, we assume results from at most one completed phase can be sold in a given period. We set $\alpha_m = \alpha \times (1 + \eta_c(m/M))$ to capture the dependence of the probability of raising contributions from ex-post donors on the number of completed phases. We vary $\alpha$ and $\eta_c$ to quantify the impact of ex-post donor characteristics on performance.

In addition to the probabilities of raising contributions from each donor type, we also vary $\gamma$, the proportion of traditional donors in the population, to study the impact of donor mix on performance.

4.2. Funds from traditional donors for implementation stage

The first part of our numerical study deals with understanding how the various donor and project characteristics affect the optimal buffer of funds to accumulate at the beginning of the implementation stage. As seen from the analysis in Section 3 and Theorem 2, the NPO faces a tradeoff when accumulating funds from traditional donors before the implementation stage. As more funds are raised for implementation, it reduces the NPO’s dependence on contributions raised during the implementation stage to keep the implementation going. However, raising more funds for implementation means the NPO has to raise a larger amount from donors upfront and this delays the start of the implementation phase.

To determine $x^*_0(y_0)$ for a given $y_0$ and combination of donor and project characteristics, the donation process was simulated and the expected utility computed for each value of $x_0 \in \{0, 1, \ldots, M\}$ by taking expectations over 10,000 sample paths. The optimal buffer $x^*_0(y_0)$ was determined by looking up the value of $x_0$ (for a given $y_0$, donor and project characteristics) for which the expected utility was the highest. Figure 1 shows $x^*_0(y_0)$ as a function of $y_0$ for different values of $\gamma$ and $M$ for a given $\beta_0$, while Figure 2 shows $x^*_0(y_0)$ as a function of $y_0$ for different values of $\gamma$ and $\beta_0$ under different scenarios for a given $M$.

In Figure 1, we see that $x^*(y_0)$ is non-decreasing in $M$ for each $y_0$. The NPO’s funding needs increases with the number of implementation phases. As a result, for the same amount of initial
funds $y_0$, the NPO finds it optimal to start the implementation stage with a higher amount of buffer as $M$ increases.

In panels 2(a)–2(b), the NPO can raise at most one unit of funds from traditional donors in each period during the implementation stage. In panels 2(c) and 2(d), the NPO can raise more than one unit of funds in a period from traditional donors during the implementation stage.

The fact that $x_0^*(y_0)$ is non-increasing in $y_0$ is intuitive and straightforward based on the under-
lying trade-offs discussed in Theorem 2. When the NPO can raise multiple units of funds in any
given period during the implementation phase (as is the case in panels 2(c)–2(d)) compared to
when it can raise at most one unit per period (as is the case in panels 2(a)–2(b)), it has lesser
need to accumulate funds before the start of the implementation stage. Further, as the probability
of raising funds in the fixed cost stage increases, the NPO finds it optimal to accumulate a larger
amount of funds at the beginning of the implementation phase, as seen by comparing Figures 2(a)
and 2(c) with Figures 2(b) and 2(d) respectively. (In Figure 2(d), we see that $x^*(1) = 2$. This
is because the expected utility is computed using simulation and the expected values from the
simulation was such that $\bar{U}(1,1,0) \geq \bar{U}(0,1,0)$. However, the difference in expected utility for $x = 2$
and $x = 0$ was statistically insignificant at $p = 0.10$.)

Figure 3 illustrates how $x_0^*(y_0)$ varies with the probability of raising funds from the ex-post
donors during the implementation phase. As we expect, and illustrated in Figures 3(a)–3(b), the
need for funds from traditional donors decreases as the probability of receiving ex-post donations increases. (The non-monotonicity in $x^*(y_0)$ with respect to $\alpha$ in Figure 3(b) is because of statistical fluctuations in the simulation results.)

![Graph](image)

(a) $\gamma = 0.2$.

(b) $\gamma = 0.5$.

**Figure 3** Impact of ex-post donor characteristics on $x^*(y_0)$ ($\beta_0 = 0.9$, $\beta_1 = 0.3$, $\eta_e = \eta_o = 0$, $M = 15$, $F = 10$, multiple units per period).

Unlike the other parameters, the impact of a change in $\gamma$ on $x^*_0(y_0)$ is not immediately obvious because a change in $\gamma$ affects the flow of funds from traditional donors both in the fixed cost stage and implementation stage in the same direction; an increase in $\gamma$ increases the probability of contributions from traditional donors in both stages. As seen from Figures 1–3, $x^*_0(y_0)$ is non-decreasing in $\gamma$, other parameters being the same. This suggests that the benefit of de-coupling the implementation from the fund raising process in the implementation stage is more for higher values of $\gamma$.

### 4.3. Total expected utility

Figures 4 and 5 show the impact of donor characteristics on the total expected utility delivered. While it is intuitive that the total expected utility is increasing in the probability of raising contributions from traditional and ex-post donors, the impact of a change in $\gamma$ is not obvious. As these figures illustrate, the total expected utility is increasing in the proportion of traditional donors.

Figure 6 illustrates how the optimal expected utility varies with $y_0$ as the number of implementation phases increases. Because each phase delivers utility, the total utility increases as $M$ increases.
Therefore, to compare the impact of the initial funds \( y_0 \) for different \( M \), it is useful to compare the normalized discounted utility, \( \bar{U}^*(y_0) \times \frac{1 - \delta}{1 - \delta M} \), rather than \( \bar{U}^*(y_0) \) directly. The normalized utility is the ratio between the expected discounted utility delivered to the utility that the NPO could have delivered if the NPO had sufficient funds to meet the project’s fixed and implementation costs.
Figure 6 Impact of donor and project characteristics on $U^*(y_0)$ \( (\beta_0 = 0.6, \beta_1 = 0.3, \alpha = 0.5, \eta_t = 0.2, \eta_c = 0, F = 10) \).
Figure 6 shows that for lower values of \( y_0 \), the normalized utility is non-increasing in \( M \) while for sufficiently large \( y_0 \), an increase in \( M \) doesn’t have a significant impact on the normalized utility delivered. This is because a large \( y_0 \) buffers the NPO in the implementation phase and makes it less dependent on the flow of donations.

4.4. Value of ex–post funding approach

Attracting ex-post donors allows the NPO to increase the donor base and thus deliver higher expected utility. However, seeking donations ex-post also exposes the NPO to a risk of shortfall because the NPO has to use its own funds to implement the project phases and recovering the funds spent is uncertain. It is therefore instructive to understand the benefit due to attracting ex-post donors and how the benefit depends on various factors. In this section, we compare the total expected utility delivered by raising contributions from both traditional and ex-post donors with the expected utility delivered when the NPO uses the traditional approach; that is, the NPO raises all the funds required to implement the project from traditional donors before the start of the implementation stage by choosing \( x_0 = M \).

Figure 7 shows the increase in total expected utility \( \bar{U}^*(y_0) - \bar{U}_{TF} \) as a percentage of \( \bar{U}^*(y_0) \), where \( \bar{U}_{TF} \) is the expected utility delivered when the NPO uses a traditional approach. There are two sources of benefit, compared to the traditional approach, when using the ex-post funding approach.

(a) \( \eta_t = 0, \eta_c = 0 \).

(b) \( \eta_t = 0.2, \eta_c = 0.2 \).

Figure 7 Effectiveness of ex–post funding approach (\( \beta_0 = 0.6, \beta_1 = 0.3, \alpha = 0.5, M = 15, F = 10, \) multiple units per period).

Figure 7 shows the increase in total expected utility \( \bar{U}^*(y_0) - \bar{U}_{TF} \) as a percentage of \( \bar{U}^*(y_0) \), where \( \bar{U}_{TF} \) is the expected utility delivered when the NPO uses a traditional approach. There are two sources of benefit, compared to the traditional approach, when using the ex-post funding approach.
approach. First, the NPO’s initial funds available during the implementation phase reduces the
dependence of the NPO on raising contributions from donors. Second, even when the NPO has no
initial funds available \((y_0 = 0)\), by strategically choosing to start project implementation before full
funds are available, the NPO can start delivering utility earlier. As seen in Figure 7, the benefit
from starting the implementation earlier allows the total expected utility delivered to be almost
twice as much as the expected utility delivered when following a traditional approach and waiting
for all funds to become available before implementation.

As the numerical results in Figure 7 show, the value of the ex-post approach is non-increasing
in \(\gamma\). This is because the availability of initial funds enables the NPO to reduce the dependence
on the flow of traditional donor contributions during the implementation stage. As the propor-
tion of traditional donors increases, the benefit of the NPO’s initial funds diminishes because the
probability of raising funds from traditional donors is higher.

Figures 8(a) and 8(b) show how the marginal value of the initial funds available with the NPO
varies with \(\gamma\) and \(\alpha\), respectively. A small value of \(\gamma\) indicates a large proportion of ex-post donors,
which implies a higher probability of contributions from ex-post donors, everything else being
equal. Any corpus that is spent is therefore recovered faster. On the other hand, when \(\gamma\) is high, the
fraction of ex-post donors in the donor population is low and any corpus spent by the NPO takes
longer to recover. Thus, for low values of \(y_0\), the marginal value of any additional corpus is limited
for extremal values of \(\gamma\), as seen from Figure 8(a). For intermediate values of \(\gamma\), having additional
corpus helps reduce the dependence on donor contributions during the implementation and thus
additional corpus is more valuable. When \(y_0\) is sufficiently high, the impact of donor contributions,
irrespective of the mix of donors in the population, on continuity of implementation is reduced.
As a result, the fraction of traditional donors in the population has lesser impact on the marginal
value of additional corpus.

As \(\alpha\) increases, the probability of a ex-post donor contributing is high. Thus, even with limited
corpus the NPO can continue implementing the various phases of the project by recovering the
funds spent faster as $\alpha$ increases. Thus, for low values of $y_0$, the marginal benefit of $y_0$ is non-decreasing in $\alpha$ as seen in Figure 8(b). For sufficiently high $y_0$, the NPO can implement a larger number of phases before the donor contributions become a constraint. As a result, an increase in $\alpha$ has a lower impact on the marginal value of initial funds available with the NPO.

4.5. Allocation of NPO’s initial funds

The analytical model and numerical studies in earlier sections assumed that the NPO’s initial funds were reserved for the implementation stage. However, given the numerical results about the impact of the fixed cost stage on the total expected utility, the NPO can be strategic about the use of its initial funds and keep only a part of it for implementation and use the remaining towards meeting some of the fixed costs. That is, suppose the NPO’s total initial funds is $R$, with $R < F + M$. The NPO can choose $y_0 \leq R$ as the funds reserved for the implementation stage and use the remaining $R - y_0$ towards the fixed cost. That is, the funds to be raised from traditional donors before the start of the implementation stage is equal to $\max\{0, F - (R - y_0)\}$, in addition to the buffer $x_0$ that the NPO would like to have at the beginning of the implementation stage. The trade-off involved
in determining the optimal $y_0$ is similar to the trade-off involved in determining $x^*_0(y_0)$: increasing $y_0$ delays starting the implementation stage but makes the implementation of the project phases less dependent on flow of contributions from donors during the implementation stage.

Figure 9 shows the total expected utility as a function of $y_0$, the amount of NPO’s initial funds allocated to the implementation phase when the NPO has total initial funds of $R = 15$ available.

As Figure 9 shows, it may not be optimal to reserve all of the NPO’s initial funds for implementation and instead use part of it towards meeting the fixed costs. In each of Figures 9(a)–9(c), the optimal allocation of funds is to meet all the fixed cost and use the remaining funds, $R - F$, for implementation. Such an allocation enables the NPO to begin implementation almost immediately (notice from Figure 2 that $x^*(y_0) = 0$ for $y_0 \geq 5$ except for $\beta_0 = 0.9$ and $\gamma = 0.9$). While we do not report the results here, the optimal allocation results are similar for other values of $R$ and $F$, with
the NPO’s initial funds being used to meet fixed costs first and then allocated for the implementation stage. These results suggest that the benefit from starting the implementation and delivering benefits is significant compared to the benefit from having the implementation of different project phases less dependent on flow of funds during the implementation stage. This is because the NPO can raise donations from both traditional and ex-post donors during the implementation stage while it can raise funds only from the traditional donors during the fixed cost stage.

While it is analytically intractable to determine the optimal allocation of initial funds between the fixed cost and implementation stages under the assumptions made in §3, the results of these numerical studies suggest that using as much of the initial funds towards the fixed cost and using the remaining funds for implementation is a good heuristic that NPOs can use to improve the value from using an ex-post funding approach. With additional assumptions that (i) the NPO does not raise funds in excess of the fixed cost $F$ during the fixed phase, and (ii) only ex-post donors contribute during the implementation phase, we are able to characterize the optimal allocation of initial funds between the two stages. For brevity, we include the details of this result in an additional appendix (see §B.1).

Finally, we study the robustness of our results by considering different probability distributions. The results of our numerical studies when probability of contributions from traditional donors follows a binomial distribution are included in the Appendix B.2. Additionally, we also report results from numerical studies where we relax the assumption that only one type of donor, traditional or ex-post, contributes in any time period during the implementation phases. It is noteworthy that the findings and insights from the analysis remain unchanged even under these conditions.

5. Discussion and Future Research

In this paper we analyze the operations of a resource constrained NPO that implements social development projects and raises funds from donors using an ex-post funding approach when the donor population consists of traditional and ex-post donors. Traditional donors contribute in anticipation of results, while ex-post donors contribute only after delivery of results. We show that
there exists an optimal amount of surplus funds that the NPO should accumulate from traditional donors before starting implementation. Using numerical studies, we demonstrate that the benefit from an ex-post funding approach can be an order of magnitude higher than the expected benefit from a traditional approach for a wide range of parameters and probability distributions that characterize donor behavior. Furthermore, even with relatively small amounts of initial funds, the NPO can deliver up to twice the benefit of traditional funding by using an ex-post funding approach. Our numerical results suggest that the relative benefit from an ex-post funding approach is non-increasing in the fraction of traditional donors.

To the best of our knowledge, this paper is amongst the first to model the operations of NPO using an ex-post funding approach to raise contributions and implement projects. Specifically, we benchmark against traditional funding model and quantify the benefits from adopting the ex-post funding model. Beyond making the NPO’s operations transparent to donors, our results suggest that there are significant operational benefits that can be realized by optimally raising funds from different donor types.

This work suggests several avenues for future research. In our analysis, we have assumed that the NPO’s utility is perfectly aligned with the benefit delivered to the target beneficiaries of the project, and as a result the NPO is interested in maximizing the total expected benefit delivered. As NPOs cannot disburse any surplus to stakeholders and therefore unlikely to have a profit motive, our modeling of the NPO’s objective function is in line with this feature of NPOs. However, it is possible that a NPO might enjoy private benefits that do not accrue to the project beneficiaries. For instance, there might be private utility that a NPO enjoys upon completion of a project, or a private cost of exerting effort to implement the project. Our model can be extended to include such private utilities without significant changes, and the essential trade-offs in the funds allocation decision remain unaltered.

We have considered the implementation of “proven” projects where the probability of failure to deliver benefits to the target population is minimal. In reality, development projects require the active participation of multiple stakeholders, including the intended beneficiaries themselves. It is
likely that sometimes the implementation of a given phase may not yield the intended benefits, and the phase can “fail”. We can extend the model in the current paper in different ways, to accommodate project failures. One approach is to assume that the probability of raising funds against a failed implementation phase is lower. While we assume the probability of selling results from a completed phase is fixed, the model can be extended to randomize the probability of raising funds from ex-post donors to capture failures during implementation. In a more general setting the NPO may have the ability to work on a portfolio of social-development projects. In such settings the NPO may have the ability to cross subsidize across projects. Additionally, the probability of raising donations may depend on the portfolio of projects undertaken and the number of phases completed successfully. These issues raise interesting questions and provide interesting avenues of future research.

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References


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Appendix A: Proofs of theoretical results

Proof of Lemma 1: When there are $\theta \leq F + x$ units of funds yet to be raised, we have

$$E[\delta^{\tau(\theta)}] = \delta \left[ \gamma \left( \beta_0 E[\delta^{\tau(\theta-1)}] + (1-\beta_0) E[\delta^{\tau(\theta)}] \right) + (1-\gamma) E[\delta^{\tau(\theta)}] \right]$$

when $\theta > 0$ and $E[\delta^{\tau(\theta)}] = 0$ for $\theta = 0$, where we use $\tau(\theta)$ to denote the time required to raise $\theta$ units of funds. Through recursive substitution, we can see that $E[\delta^{\tau(\theta)}] = \eta^\theta$ where $\eta = \frac{\delta \beta_0}{1-\delta \beta_0 + (1-\beta_0) \gamma}$. The proof is complete because $\tau_F(x) = \tau(F + x)$ and therefore $\bar{\delta}(x) = E[\delta^{\tau(F+x)}] = \eta^{F+x}$.

Proof of Theorem 1: The proof is by induction. Consider the case $m = M - 1$. From equation (6), we have $U(x, y, M - 1) = u$ for all $x \geq 0$ when $y > 0$ and therefore $\Delta_x(y, M - 1) = 0$ for all $x \geq 0$.

For $y = 0$, we have $U(x, 0, M - 1) = u$ for $x \geq 1$ and $\Delta_x(0, M - 1) = 0$ for $x \geq 1$.

It is easy to see that $\Delta_{x+1}(y, M - 1) = 0 = \Delta_{x+1}(y, M - 1)$. Suppose $\Delta_x(y, m) \leq \Delta_{x-1}(y, m)$ for all $x = 2, \ldots, m$, and $\Delta_{x+1}(y, m + 1) \leq \Delta_{x}(y, m + 1)$ for all $y$.

When $y > 0$, we have

$$\Delta_{x+1}(y, m) = U(x + 2, y, m) - U(x + 1, y, m)$$

$$= \delta \left[ \gamma \left( \beta_{m+1} U(x + 2, y, m + 1) - U(x + 1, y, m + 1) \right) + (1-\beta_{m+1}) \left( U(x + 1, y, m + 1) - U(x, y, m + 1) \right) \right]$$

$$+ (1-\gamma) \left[ \alpha_{m+1} U(x + 1, y, m + 1) - U(x, y, m + 1) \right]$$

$$+ (1-\alpha_{m+1}) \left( U(x + 1, y, m + 1) - U(x, y, m + 1) \right) \right]$$

$$= \delta \left[ \gamma \left( \beta_{m+1} \Delta_{x+1}(y, m + 1) + (1-\beta_{m+1}) \Delta_x(y, m + 1) \right) + (1-\gamma) \left[ \alpha_{m+1} \Delta_x(y + 1, m + 1) + (1-\alpha_{m+1}) \Delta_x(y, m + 1) \right] \right]$$

$$+ (1-\gamma) \left[ \alpha_{m+1} \Delta_x(y + 1, m + 1) + (1-\alpha_{m+1}) \Delta_x(y, m + 1) \right]$$

$$\leq \delta \left( \gamma \beta_{m+1} \Delta_x(y, m + 1) + (1-\beta_{m+1}) \Delta_{x-1}(y, m + 1) \right)$$

$$+ (1-\gamma) \left( \alpha_{m+1} \Delta_{x-1}(y + 1, m + 1) + (1-\alpha_{m+1}) \Delta_{x-1}(y, m + 1) \right)$$

$$= \Delta_x(y, m)$$

where the inequality follows from the induction hypothesis. Thus the theorem is true for $y > 0$. We can show that the theorem is true even when $y = 0$ by following the same steps as above.
Proof of Theorem 2: We have

\[ \bar{U}(x+1, y_0) - \bar{U}(x, y_0) = \delta(x+1)U(x+1, y_0, 0) - \delta(x)U(x, y_0, 0) \]

\[ = \eta^{F+x+1}U(x+1, y_0, 0) - \eta^{F+x}U(x, y_0, 0) \]

\[ = \eta^{F+x}[\eta U(x+1, y_0, 0) - U(x, y_0, 0)] \]

\[ = \eta^{F+x}[\eta(U(x+1, y_0, 0) - U(x, y_0, 0)) - (1 - \eta)U(x, y_0, 0)] \]

\[ = \eta^{F+x+1}U(x, y_0, 0) \left[ \frac{U(x+1, y_0, 0) - U(x, y_0, 0)}{U(x, y_0, 0)} - 1 - \eta \right] \]

\[ = \eta^{F+x+1}U(x, y_0, 0) \left[ \frac{\Delta_x(y_0, 0)}{U(x, y_0, 0)} - \frac{1 - \eta}{\eta} \right] \]

From Theorem 1, we know that \( \Delta_x(y_0, 0) \) is non-increasing in \( x \) for \( x \geq 1 \) and furthermore \( U(x, y_0, 0) \) is non-decreasing in \( x \). Thus, \( \frac{\Delta_x(y_0, 0)}{U(x, y_0, 0)} \) is non-increasing in \( x \). Therefore, there exists a unique \( \hat{x} \geq 1 \) such that

\[ \frac{\Delta_x(y_0, 0)}{U(x, y_0, 0)} \geq \frac{1 - \eta}{\eta} \text{ for } x < \hat{x} \]

\[ \frac{\Delta_x(y_0, 0)}{U(x, y_0, 0)} < \frac{1 - \eta}{\eta} \text{ for } x \geq \hat{x} \]

Thus, \( \bar{U}(x, y_0) \) is non-decreasing in \( x \) for \( 1 \leq x < \hat{x} \) and decreasing in \( x \) for \( x \geq \hat{x} \).

If \( \hat{x} = 1 \), then \( x^* = 1 \) if

\[ \bar{U}(1, y_0) \geq U(0, y_0) \Leftrightarrow \delta(1)U(1, y_0, 0) \geq \delta(0)U(0, y_0, 0) \]

\[ \Leftrightarrow \eta^{F+1}U(1, y_0, 0) \geq \eta^F U(0, y_0, 0) \]

\[ \Leftrightarrow \frac{U(1, y_0, 0) - U(0, y_0, 0)}{U(0, y_0, 0)} \geq \frac{1 - \eta}{\eta} \]

and \( x^* = 0 \) otherwise.

Proof of Theorem 3: The proof is by induction. The induction steps are similar to those in the proof of Theorem 1 and we skip them here for brevity.
Appendix B: Additional Appendix

B.1. Optimal allocation of initial funds

In §4.5, we used numerical experiments to explore how a resource constrained NPO can benefit by allocating its initial funds between the fixed cost and implementation stages judiciously. In this appendix, we develop analytical results on how the NPO should allocate its funds $R$ between the fixed cost and implementation stages under some additional assumptions regarding the donor contributions.

Let the NPO reserve $y_0 \leq R$ of its initial funds for the implementation phase, with the remaining $R - y_0$ being used to meet part of the fixed costs and suppose

Assumption 1. 1. The traditional donors do not contribute during the implementation stage. That is, for all $m > 0$ and $j > 0$, we have $p_j(m) = 0$.

2. During the fixed cost stage, the NPO only raises as much funds as is required to meet the fixed costs. That is, $x_0 = 0$ and $\theta = F - (R - y_0)$ for all $y_0$.

With Assumption 1, we have $\bar{U}(0) = E\left[\delta^{F-(R-y_0)}U(y_0,0)\right] = \left[\frac{\delta \gamma \beta_0}{1 - \delta \gamma (1 - \beta_0) + 1 - \gamma}\right]^{(F-(R-y_0))^+}$. Also, because $x_0$ is zero and $p_j(m) = 0$ for all $j$ and $m$ greater than zero, we can omit $x$ from the state variable and simply write $U(x, y, m)$ as $U(y, m)$ for all $y$ and $m$. The expected utility $U(y, m)$ given by equation (6) can be simplified and written as

$$U(y, m) = \begin{cases} 
    u + \delta [(1 - \gamma)\alpha_{m+1}U(y, m+1) + ((1 - \gamma)(1 - \alpha_{m+1}) + \gamma)U(y - 1, m + 1)] & \text{if } y_0 \leq y > 0 \text{ and } m < M, \\
    \delta [(1 - \gamma)\alpha_mU(y + 1, m) + ((1 - \gamma)(1 - \alpha_m) + \gamma)U(x, y, m)] & \text{if } y = 0 \text{ and } m < M, \\
    0 & \text{if } m = M.
\end{cases}$$

(B.1)

Using the same arguments as those used to prove Theorem 1, we can show that

Theorem B.1. The expected discounted benefit $U(y, m)$ is non-decreasing in $y$ for each $m$. Further, $U(y + 2, m) - U(y + 1, m) \leq U(y + 1, m) - U(y, m)$ for all $y \geq 0$, for all $m$.

An immediate implication of Theorem B.1 is that $U(y_0, 0)$ is concave in $y_0$, the funds allocated for the implementation phase. Thus, additional funds reserved for the implementation phases have diminishing returns and it may not be optimal to reserve the entire endowment for the implementation phase. Theorem B.2 shows that the total expected utility $\bar{U}(y_0) = E\left[\delta^{F-(R-y_0)}U(y_0,0)\right]$ is unimodal in $y_0$ and characterizes the optimal allocation of funds, $y_0$, to reserve for the implementation stage.
Theorem B.2. For an NPO using an ex-post funding approach, there exists a unique optimal amount of funds, $y_0^*$, to allocate for the implementation phase.

Proof of Theorem B.2: Let $\eta = \frac{\delta y_0}{1 - [\gamma(1 - y_0) + 1 - \eta]}$. We thus have $\bar{\delta}(0) = \eta^{(F - (R - y_0))^+}$. For any $y_0$ such that $F \geq R - y_0$, we have

$$
\bar{U}(y_0 + 1) - \bar{U}(y_0) = \eta^{(F - (R - y_0 - 1))}U(y_0 + 1, 0) - \eta^{(F - (R - y_0))}U(y_0, 0)
$$

$$
= \eta^{(F - (R - y_0))} \left[ \eta U(y_0 + 1, 0) - U(y_0, 0) \right]
$$

$$
= \eta^{(F - (R - y_0))} \left[ \eta \left( U(y_0 + 1, 0) - U(y_0, 0) \right) - U(y_0, 0)[1 - \eta] \right]
$$

$$
= \eta^{(F - (R - y_0))} U(y_0, 0) \left[ \frac{U(y_0 + 1, 0) - U(y_0, 0)}{U(y_0, 0)} - \frac{1 - \eta}{\eta} \right]
$$

From Theorem B.1, we know that $U(y_0, 0)$ is concave and increasing in $y_0$. Therefore, the first term inside the parentheses is non-negative and decreasing in $y_0$. Thus, there exists a unique $\hat{y}_0$ such that

$$
\frac{U(y_0 + 1, 0) - U(y_0, 0)}{U(y_0, 0)} \geq \frac{1 - \eta}{\eta} \text{ for } y_0 < \hat{y}_0
$$

$$
\frac{U(y_0 + 1, 0) - U(y_0, 0)}{U(y_0, 0)} < \frac{1 - \eta}{\eta} \text{ for } y_0 \geq \hat{y}_0
$$

Thus, $\bar{U}(y_0)$ is non-decreasing in $y_0$ for $1 \leq y_0 < \hat{y}_0$ and decreasing for $y_0 \geq \hat{y}_0$ and it is optimal for the NPO allocate $\hat{y}_0$ for the implementation stage and $R - \hat{y}_0$ to the fixed cost stage if $R - \hat{y}_0 \leq F$. On the other hand, if $\hat{y}_0$ is such that $R - \hat{y}_0 > F$, it is optimal for the NPO to allocate $F$ to the fixed cost stage and $R - F$ to the implementation stage.

\[\blacksquare\]
B.2. Alternate probability distribution for traditional donor contributions

In §4.2–4.4, the contributions from traditional donors were assumed to follow a probability distribution with constant failure rate; that is, for each \( m \), \( \frac{p_j(m)}{1 - \sum_{k=0}^{j} p_k(m)} \) was a constant that did not vary with \( j \). In this section, we consider an alternate probability distribution for contributions from traditional donors such that the failure rate is not constant. Specifically, the contributions from traditional donors during each period in the fixed cost stage follow a binomial distribution \( B(F + M, \beta_0) \), while the contributions during the implementation stage when the NPO has \( x \) units of funds from traditional donors available and \( m \) phases completed follows a binomial distribution \( B(M - m - x, \beta_m) \).

Figures B.1–B.3 illustrate how \( x^*(y_0), \bar{U}^*(y_0) \) and benefit from the ex-post funding approach vary with \( y_0 \) when the contributions from traditional donors follow a binomial distribution.

![Figure B.1](image)

(a) \( \beta_0 = 0.3 \).

(b) \( \beta_0 = 0.6 \).

Figure B.1  **Impact of traditional donor characteristics on** \( x^*(y_0) \) \((\beta_1 = 0.3, \alpha = 0.5, \eta_t = \eta_c = 0, M = 15, F = 10, \) binomial distribution).**

As Figures B.1–B.3 show, the observations from the earlier numerical study results continue to hold with this different probability distribution as well.

B.3. Traditional and Ex-post donors contributing in a time period

In the analysis and numerical studies thus far, it was assumed that the NPO could receive contributions from only one type of donor in any given period during the implementation stage; that is, either traditional or ex-post donors contributed in a given period, but not both. In this section, we relax this assumption to consider the possibility of both types of donors can contribute in a given period. Specifically, \( \gamma \) models the probability that a traditional donor contributes in a given period, with the probability distribution of contribution from the traditional donor given by \( p_j(x, m) \) as specified in § 4.1. Conditional on a traditional
donor contributing in a period, $\theta_0$ is the probability that an ex-post donor also contributes in the same period. That is, $\gamma \times \theta_0$ is the probability that both traditional and ex-post donors contribute in a given period and $\gamma \times (1 - \theta_0)$ is the probability that only a traditional donor contributes in the period. Conversely, when a traditional donor does not contribute in a given period, $\theta_1$ is the probability that an ex-post donor contributes in the same period. That is, $(1 - \gamma) \times \theta_1$ is the probability that only an ex-post donor contributes in the given period. The numerical study in § 4 is a special case where $\theta_0 = 0$ and $\theta_1 = 1$.

Figures B.4 and B.5 illustrate how $x^*(y_0)$ and $\bar{U}^*(y_0)$ vary with $y_0$ for different values of $\theta_0$ and $\theta_1$. As these illustrations show, the observations from the earlier numerical study results continue to hold even if we allow both types of donors to contribute in a given period.
(a) $\theta_0 = 0.25$, $\theta_1 = 0.5$.

Figure B.4  Impact of donor characteristics on $x^*(y_0)$ ($\beta_0 = 0.9$, $\beta_1 = 0.3$, $\alpha = 0.5$, $\eta = \eta_c = 0$, $M = 15$, $F = 10$)

(b) $\theta_0 = 0.5$, $\theta_1 = 1$.

Figure B.5  Impact of donor characteristics on $\bar{U}^*(y_0)$ ($\beta_0 = 0.9$, $\beta_1 = 0.3$, $\alpha = 0.5$, $\eta = \eta_c = 0$, $M = 15$, $F = 10$)