Abstract
Recently, the development of ransomware strains as well as changes in the marketplace for malware have greatly reduced the entry barrier for attackers to conduct large-scale ransomware attacks. In this paper, we examine how this mode of cyberattack impacts software vendors and consumer behavior. When victims face an added option to mitigate losses via a ransom payment, both the equilibrium market size and the vendor’s profit under optimal pricing can actually increase in the ransom demand as well as the risk of residual losses following a ransom payment (which reflect the trustworthiness of the ransomware operator). We further show that for intermediate levels of risk of the vulnerability being successfully exploited, the vendor restricts software adoption by substantially hiking prices. This lies in stark contrast to outcomes in a benchmark case involving traditional malware (non-ransomware) where the vendor will choose to decrease price as security risk increases. Social welfare is higher under ransomware compared to the benchmark in both sufficiently low and high risk settings. However, for intermediate risk, it is better from a social standpoint if consumers do not to have an option to pay ransom. We also show that the expected total ransom paid is non-monotone in the risk of success of the attack, increasing when the risk is moderate in spite of a decreasing ransom-paying population. For ransomware attacks on other vectors (as opposed to patchable vulnerabilities), there can still be incentives to hike price. However, market size and profits instead weakly decrease in the ransom amount, and strategic discontinuous reductions in price due to increased risk are no longer observed. When studying a generalized model that includes both traditional and ransomware attacks, our results remain robust to a wide range of scenarios, including threat landscapes where ransomware has only a small presence.
1 Introduction

In recent years, ransomware has evolved to become a prevalent class of malware due to improved use of encryption and attack vectors as well as increased maturity of cryptocurrency-based payment systems (such as Bitcoin and Ethereum exchanges) which obfuscate via pseudonymity the identity of transacting parties (Verizon 2018). Ransomware is an extortion-based attack that infects a computer system and subsequently prevents either access to the system (i.e., locker ransomware) or access to files or data (i.e., crypto ransomware) (Savage et al. 2015). Victims are typically threatened with permanent loss of access unless they pay a ransom. Having an additional decision for users (i.e., whether to pay ransom) disrupts the economics underlying software usage and patching behaviors, and therefore ransomware may necessitate management strategies and policies that conflict with what served prior environmental characteristics well.

Over the past decade, ransomware experienced tremendous growth and even held the crown as the fastest growing cybersecurity threat (Cybersecurity Insiders 2017). According to Malwarebytes (2017), six out of ten malware payloads were ransomware in the first quarter of 2017. The number of ransomware variants increased 4.3 times between the first quarter of 2016 and the first quarter of 2017 (Proofpoint 2017). The rate of ransomware attacks on businesses has been accelerating from one attack every 40 seconds in 2016 to one attack every 14 seconds in 2019 (Kaspersky 2016, Morgan 2019). In a survey of 5,000 IT managers across 26 countries, Sophos found that over 50% were hit by ransomware last year and attacks were successful in 73% of the cases (Sophos 2020). Moreover, ransomware has continued to evolve to cause increased downtime, now averaging over 16 days (Palmer 2020). The overall damage that businesses incur from ransomware attacks (including remediation and lost business) is expected to reach $20 billion by 2021, a whopping 57 times increase compared to 2015 (Morgan 2019).

Because of ransomware’s prevalence, businesses and users have now had experiences with the threat and importantly begun to put strategies and policies in place for managing it going forward (Ali 2016; Davis 2018; Mercer 2018). However, managing cybersecurity is a difficult task because the threat landscape can fluctuate significantly year to year driven by changing tides and the wide-ranging motivations of hackers. Preventative actions are the best defense against ransomware (FBI 2016, U.S. Department of Justice 2017, No More Ransomware Project 2017). In fact, the U.S. Department of Health and Human Services delineates what healthcare providers are required to do to prevent ransomware infection in order to be HIPAA compliant (U.S. Department of Health and Human Services 2016). Timely patching of systems, as well as responsible user access and communication management are considered
among best practices, but many organizations and users regrettably do not comply. Sadly, this state of affairs has been the defining characteristic of security vulnerabilities for decades, and ransomware similarly exploits the same poor risk management practices.

Consider the example of WannaCry, one of the most prevalent ransomware attacks observed in the last few years which leveraged NSA-leaked infiltration and exploit tools (Sanders 2019). Microsoft had released a patch on March 14, 2017, following revelation of the vulnerability’s existence by The Shadow Brokers hacker group (Microsoft 2017b). Two months later, despite Microsoft having made the patch available, a sizeable number of unpatched systems enabled WannaCry to spread laterally fast, indiscriminately affecting over 230,000 computers across 150 countries in a day (Cooper 2018). Even one month after global news outlets alerted the world to the vulnerability being exploited by WannaCry, many users and organizations had still not patched and, as a result, the more destructive NotPetya malware was able to spread using the same vulnerability (Microsoft 2017a). Fast forwarding to a couple of years after initial appearance, during the entire 2019 and in Q1 2020, WannaCry still accounted for the highest number of detected ransomware attacks (Kaspersky 2020).\footnote{While the initial WannaCry attack was partially thwarted due to a kill switch associated with the malware that halted spread, within days WannaCry variants without the kill switch started spreading again (Sanders 2019).}

In fact, in the first half of 2019, WannaCry attacks alone accounted for over 6 times as many detections compared to attacks from all other ransomware variants \textit{combined}; during that time, it achieved 3500 successful infections per hour and an estimated 145,000 infected devices across 103 countries that continued to support its self-propagation (Sanders 2019, Trend Micro Research 2019). These incidents highlight how large populations of unpatched users encourage the development of ransomware, facilitate its spread, and also keep the threat current. As software vendors and government agencies grapple with the significant losses being incurred, they have sought to understand how to respond to and operate in this new environment where affected users and organizations now face a ransom demand that can possibly mitigate losses.

Large-scale ransomware campaigns can also spread via unpatchable vectors such as phishing attacks and zero-day vulnerabilities. Phishing attacks bait users into action (e.g., opening attachments laced with malware, clicking on fake banner ads, and clicking on malicious URLs designed to siphon credentials or deliver a drive-by download). As an example, in August and September of 2017, Locky ransomware was pushed via multiple massive phishing campaigns to millions of users (Cabuhat et al. 2017; Palmer 2017), exploiting a long known Microsoft Office vulnerability that Microsoft only permanently disabled in December 2017. In another example in 2019, attackers exploited a zero-day vulnerability (since patched) in the widely
used Oracle WebLogic server to install Sodinokibi and GandCrab ransomware on vulnerable machines, which necessitated no user interaction at all (Goddin 2019); this is among the first known cases where bad actors used a single attack to distribute two ransomware payloads (Splinters 2019).

Users facing large-scale cyberattacks, including ransomware, are also exposed to interdependent security risks. A larger at-risk population increases the risk for all individuals within the population. This can happen through a variety of mechanisms. Ransomware worms, such as ZCryptor, WannaCry, or Bad Rabbit, have the ability to self-replicate and travel laterally from an infected system to other unprotected systems on the same computer network without any additional interaction or hacker intervention (Barkly 2017). In other instances, hackers install scanners on compromised systems to harvest credentials that enable them to infiltrate additional endpoints within the same corporation as well as within the networks of business partners and clients (Barak 2020). Lastly, having more users at risk can attract increased attention from malicious hackers; this, in turn, can also translate into higher security risk indirectly. In general, the risk of ransomware infection is likely to be characterized by network externalities.

Hacker motivations span human curiosity, a desire for fame, an anti-establishment agenda, economic objectives, hacktivism, and even cyberwarfare (Thomas and Stoddard 2012). Both NotPetya and WannaCry, the recent and largest ransomware attacks in history, were attributed to state actors, i.e., Russia and North Korea, respectively (Chappell and Neuman 2017; Marsh 2018). In the case of SamSam ransomware, two Iranian nationals were indicted for their involvement; these hackers seemed to be more economically motivated and earned $6 million in ransom payments (Barrett 2018). Similarly, CryptoLocker is speculated to have generated over $30 million in ransom payments in 100 days (Jeffers 2013).

In that state actors’ motivations are typically political in nature, those responsible for NotPetya and WannaCry did not bother to properly set up and configure effective processes to receive payments and return decryption keys to those who paid (Greenberg 2018). As of Dec 2019, only 430 WannaCry victims paid the ransom demand (WebTitan 2019). Despite not having that intent, they clearly proved the feasibility of launching large-scale and disruptive ransomware attacks. An interesting question is whether these attacks could have caused greater economic damages (and been even more successful from a malicious perspective) had the ransom payment and decryption key delivery process actually been functional. For SamSam and CryptoLocker, which were clearly motivated by revenue generation, an open question is how would the scaling of such attacks to harness the interdependent risk characteristics of large-scale attacks impact revenues, considering that business and end users would adjust their patching and usage strategies to such threats in expectation. It is easy
to see that the actual potential of ransomware has yet to be observed, and the economic models we develop in this paper help provide insight into what might lie on the horizon.

Most prior work that has aimed at understanding how a software firm and its users react to security risk tend to model both patching costs and security losses, and these models can cover a wide variety of cyber attacks (August and Tunca 2006; Cavusoglu et al. 2008; Dey et al. 2015). However, ransomware presents users with an opportunity to pay a fee in exchange for a possible reduction in security losses which alters the balance of economic factors. To explore the impact of this cybersecurity threat, we construct a series of models that include the primitive elements that uniquely define ransomware. We then examine how the threat of ransomware affects consumers’ (both businesses and individuals) choices as they face trade-offs between ex-ante security protection efforts like patching and ex-post ransom payments to agents with unlawful motives. As a class of attacks, ransomware presents a potential efficiency gain by offering a loss-mitigating payment opportunity, whereas in models of traditional attacks, victims typically do not have this opportunity and instead incur large valuation-dependent losses. On the other hand, such shifting of consumer incentives and their strategies modifies the network externality stemming from at-risk usage which fundamentally alters a vendor’s incentives and the decision problem that he faces. These trade-offs become even more complex when both ransomware and traditional attacks are commingled in a single framework that sheds light on the reach of ransomware’s impact to firm strategy. In totality, we seek to understand how ransomware characteristics affect software pricing, usage and security in the presence of interdependent risk, and reflect on whether a shift in attack trends toward increased representation from the ransomware class is helpful or hurtful to the economy.

2 Literature Review

This work contributes to several research streams falling under the general topic of economics of information security, namely (i) economics of ransomware, (ii) network security externalities due to interdependent risks, and (iii) disaster recovery. Moreover, due to the peculiarities of ransomware attacks, this work is directly related to the research stream on (iv) economic dynamics of hostage taking and negotiation.

Ransomware attacks are perpetrated based on the concept of holding hostage a digital asset and demanding a ransom for its release (Young and Yung 1996). There exists an established research stream on hostage taking, ensuing negotiations, and outcomes in scenarios involving human victims. Several empirical studies explore the effect of deterrence policies and concession making on recurrence of hijacking events (Brandt and Sandler 2009, Brandt
et al. 2016) and factors impacting the attackers’ perpetration and negotiation effectiveness (Gaibulloev and Sandler 2009). Other studies take a behavioral approach to understand terrorist actions in hostage-taking events (e.g., Wilson 2000). Early game-theoretical studies on this topic focus on the dynamics of the interaction between rational terrorists and negotiators on the part of victims (governments, families, or other interested parties). Lapan and Sandler (1988) look at multi-period scenarios where the terrorists are considering an attack each period and there are potential reputation effects propagating through time, based on government concessions during negotiations in prior attacks. They abstract the number of victims and their model characterizes attack outcomes as constant regardless of how many victims are affected. Selten (1988) explores an extension with multiple attackers and victims but each instance of an attack represents a game with an isolated outcome in which the attacker will proceed with attacking each victim separately and only if he expects some benefit from the attack.

Drawing parallels to cyberattacks, such modeling approaches can be used to characterize attacks that are to some extent isolated (small-scale) and targeted. In contrast, in the case of large-scale attacks (and in particular ransomware), the brunt of the impact is due to security interdependence as discussed in the Introduction.\footnote{This can be true even if the onset of the attack is targeted.} In these attacks, perpetrators move laterally across at-risk systems, oftentimes in an untargeted way potentially accelerated through the implementation of worm-like self-propagation; in any case, the attacker need not work through a process of decision-making for every potential breach. Observationally, in several of these attacks, the ransom demanded is hardcoded a priori to a default level rather than being adjusted based on the value of the compromised digital asset to the consumer.\footnote{WannaCry, Bad Rabbit, and ZCryptor were prompting victims to pay $300-$500, 0.05 BTC, and 1.2 BTC per affected system, respectively. Certain version of Locky were prompting users to pay 0.25 BTC.} Furthermore, theoretical kidnapping models usually involve dynamics between two parties (negotiators and attackers). In contrast, many cyberattacks are enabled by vulnerabilities in an information system sold by a legitimate vendor (developer). The vendor is partially responsible for how secure his product is and can strategically create financial incentives for the adoption and patching of the system by consumers. Our framework accommodates attacks having interdependent security risks (modeled as a negative network externality), that are conducted at scale, and we also include the role of the vendor in influencing the size of the consumer population that is vulnerable to the attack. Beyond the existence (Young and Yung 1996) and observation of cryptovirological attacks, our work focuses on their impact on software markets and the economic incentives that govern their efficacy.

The research agenda on the economics of information security has been extensively de-

On the other hand, the focus of cybersecurity recovery is often on planning and business continuity (see Bartock et al. (2016) from the National Institute of Standards and Technology (NIST) and IBM (2014)). The academic literature that explores economics of cybersecurity recovery is currently relatively sparse. We highlight how our work contributes to this nascent area. Chen et al. (2017) formalizes an ex-post recovery decision in the context of an infrastructure game where the designer can create redundant links for protection or add links back to the network post-attack as a means to recovery. In their model, whether and to what extent to heal the network is a recovery decision that must be made. Yang et al. (2019) consider a model with an advanced persistent threat (APT) where organizations attempt to mitigate the impact of APT via a dynamic quarantine and recovery (QAR) scheme. In APT settings, the timing of an attack event is necessarily more opaque, hence security actions tend to be governed by an optimal control problem specifying both a quarantine cost function and a recovery cost function. In a recent work that is the closest to ours, Cartwright et al. (2019) formally study the tradeoff between exerting ex ante costly effort to avoid an attack versus exerting ex post costly effort to recover. Their experiment assesses the impact of framing effects (as established in a rich behavioral economics literature) on this security trade-off. Notably, their study is motivated by ransomware where paying ransom is a means of recovery. The contribution of our paper is similarly more general being the first to examine a downstream endogenous recovery decision that influences an upstream security decision (i.e., patching) where these behaviors fundamentally alter the risk all agents face due to network externalities.

The study of the economic dynamics of markets affected by ransomware also remains relatively scarce. Different from other types of cyberattacks where the full loss is realized if the attack is successful, ransomware attacks present victims with a post-attack choice
(and, in some cases, opportunity to negotiate): pay ransom (and hopefully retrieve access to
the locked resource) or incur the full losses associated with giving up on that digital asset.
From the perspective of consumers, the game is more complex. Laszka et al. (2017) explore
security investments in risk mitigation (e.g., backups) and the strategic decision of whether to
pay ransom. They abstract away from preventive effort investments by consumers (patching,
firewalls, etc). In their study, the attacker’s effort is customized to the victim, thus matching
the dynamics of targeted attacks. In our study, in the case of large-scale, untargeted attacks
with risk interdependence, preventive actions effectively lessen the spreading of the attack.
Cartwright et al. (2019) adapt models by Lapan and Sandler (1988) and Selten (1988) to
ransomware attacks and explore bargaining and deterrence strategies. In particular, they
show that the likelihood of irrational aggression in the absence of payment and credible
commitment to return files upon receipt of payment play key roles in incentivizing victims
to pay the ransom. Both Laszka et al. (2017) and Cartwright et al. (2019) consider the
bargaining nature of the ransom game, where the victims have the ability to propose a
counter-offer to the demanded ransom and engage in negotiations. Again, such a modeling
approach is more relevant to targeted attacks on a smaller scale, where the effort is minimal
on the side of the attacker to customize his handling of each victim. As mentioned above,
many larger scale untargeted ransomware attacks do not allow for bargaining as the ransom
is fixed and possibly hardcoded prior to the attack taking place. Hence, in our study, we
focus more on the consumer decision of whether to pay the ransom or not in the absence of
a bargaining option.

Similar to our study, Cartwright and Cartwright (2019) and Li and Liao (2020) study
untargeted ransomware attacks with no option for bargaining. In the former paper, the
authors consider a repeated infinite-horizon game where a malicious agent attacks a randomly
chosen victim each period. In the latter, in a setting with multiple victims, the focus is on
hackers potentially engaging in an additional harmful action, that of selling victims data.
Both papers consider the role of attacker reputation given its impact on victim response and
overall payoff. In contrast, our results are consequentially impacted by the consumer usage,
protection, and ransom decisions that define relevant consumer segments in equilibrium,
and together give rise to overall market risk. We explore the strategic pricing decisions of
software vendors as well as the welfare implications associated with ransomware in such a
context.

Finally, neither the hostage-taking literature nor the extant literature on economics of
ransomware capture the possibility of negative security network externalities which often
characterize large-scale cyberattacks. Cartwright et al. (2019) mention potential spillover
effects of deterrence when there are two customer categories, but it is important to materially
tie these effects to the size of the vulnerable population. Interdependent security risks have been explored in several other papers (e.g., Kunreuther and Heal 2003, Gal-Or and Ghose 2005, Choi et al. 2010, Johnson et al. 2010, August and Tunca 2011, Hui et al. 2012, Zhao et al. 2013, Cezar et al. 2017, Hausken 2017). We extend this literature by considering risk interdependencies in the context of ransomware.

3 Ransomware Attacks on Patchable Vulnerabilities

We begin our study by focusing on classes of ransomware risk spread via patchable vulnerabilities (e.g., WannaCry). In Section 4, we then examine classes of ransomware attacks that spread via unpatchable vectors (e.g., phishing scams, zero-day vulnerabilities). Then, in Section 5, we bring other types of traditional, non-ransomware attacks into the model to demonstrate how even a limited amount of ransomware can greatly affect the strategies of software firms and social welfare.

3.1 Model Description

We study the market for a software product that exhibits security vulnerabilities exploitable by ransomware attacks. We assume a unit-mass continuum of consumers whose valuations \( v \) for the software lie uniformly on \( V = [0, 1] \).\(^4\) Each consumer makes a decision to buy, \( B \), or not buy, \( NB \). Consumers who purchase pay a price \( p \) set by the vendor for the product. When a security vulnerability arises, the vendor develops a security patch and makes it freely available to all users of the software.\(^5\) Each purchasing consumer makes a decision to patch, \( P \), or not patch, \( NP \). Consumers who decide to patch do so in a timely manner thereby incurring an expected patching cost of \( c_p > 0 \). Consumers who either do not patch or delay patching beyond a critical window face the risk of being hit by an attack. If operating unpatched, then a consumer gets hit with aggregated probability \( \pi_r u \), where \( \pi_r > 0 \) is the probability the vulnerability is exploited and \( u \) is the size of the unpatched population of users (which is endogenous to the model). Using this network externality specification, we capture the interdependent security risk associated with ransomware attacks (as discussed in Section 1).

It is worth noting that a single decision maker (such as a corporate IT department)

\(^4\)We use the term consumer generically throughout the paper, referring to both personal and corporate users.

\(^5\)For software that is currently within its support period, the norm in the software industry is to make security patches widely available for free to all users (even pirates) in order to reduce risk, a policy that is cognizant of the security externalities that exist.
who derives independent valuations from multiple systems can make separate (and possibly different) purchasing and patching decisions for each system. For example, Boeing took such a granular approach to patching for WannaCry; this led to some systems in its Commercial Airplanes division becoming affected by this ransomware because they were still unpatched almost a year after WannaCry had emerged (Gates 2018). In case like these, the aggregate valuation to a corporation is simply the sum of the individual system valuations. If the role of the system is specifically to support an individual corporate user, decision rights can be delegated to the user. The only model requirement is that the decision maker only manages a countable set of systems.

We focus on large-scale ransomware attacks. If successfully attacked, the consumer can either pay the ransom, $R$, or choose not to pay ransom, $NR$. The representative ransomware operator demands a single ransom $R > 0$ across all victims, which is consistent with many large-scale untargeted ransomware attacks in this family, including WannaCry, ZCryptor, and Bad Rabbit (Symantec 2016, F-Secure 2016, Barkly 2017). A ransomware victim with type $v$ who chooses not to pay ransom incurs losses of $\alpha v$, where $\alpha > 0$. The parameter $\alpha$ can capture a wide range of loss scenarios. First, there are operational and recovery related losses. For software that drives systems that can easily be backed up and re-deployed with minimal downtime and disruption, $\alpha$ can be small. On the other hand, for systems characterized by more intermittent back-ups or even their absence, $\alpha$ will be relatively larger. Recovery efforts might include hiring external IT security providers to perform a forensic analysis of the attack and attempt to retrieve some of the encrypted data without negotiating with attackers. Second, there can be potential losses associated with reputation, trust, goodwill, future business, and sensitive consumer data/privacy violations. For example, attackers using Maze ransomware threatened those who did not pay ransom with publication of stolen data on the Internet, and even followed through on that promise in some cases (Krebs on Security 2019). As a result, losses resulting from an attack can even go beyond the direct loss of usage and attain higher levels (i.e., $\alpha > 1$).

Even consumers who pay ransom face a risk that the attacker may not release a working decryption key. According to Sussman (2020), 32% of the victims who pay ransom do not immediately regain access to their data, and 22% never do. This can happen for multiple

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6 Ransomware financially impacts both businesses and individual users. Huang et al. (2018) tracked victims, ransomware operators and ransomware payments using end-to-end measurement, accounting for more than $16 million in payments by 19,750 likely victims over 2 years. They found that 74.5% of the infected IP addresses were residential whereas 3.5% were businesses at that time. The remaining share was composed of colleges, hosting and others.

7 However, expected losses are necessarily less than the user’s valuation. The user only makes trade-offs between paying ransom and incurring losses at the last stage of the game, consistent with the sub-game perfect equilibrium solution concept we employ.
reasons that interact with the wide-ranging attacker motivations discussed in Section 1. For example, an economically-motivated attacker may not release the key because he aims to extract more out of the victims (Siwicki 2016). Firm survey data suggests that as many as 10% of firms that pay an initial ransom are demanded a second ransom (Sussman 2020). Or perhaps not releasing decryption keys is a result of unintentional failures in either a manual process for producing and releasing keys or in the systems that process ransom payments (Abrams 2016; Chuang 2016; Chuang 2018). Attackers with either political motivations or other motivations less economic in nature may not have any intention to produce or release the keys in the first place (Frenkel et al. 2017, Marsh 2018).

Because users face uncertain ransomware risks based on uncertain motivations, we parameterize the primary loss characteristics faced by users which presents the ability to analyze outcomes across the varied motivations that underlie hacker activity. In particular, users who pay ransom still incur some residual valuation-dependent losses in expectation. We model them as scaled losses by a factor $\delta \in [0, 1]$. For example, $\delta = 1$ represents the case where the ransomware operator has no intention of releasing the decryption keys upon payment, and a smaller $\delta$ represents the opposite case where the operator uses well-functioning, automated decryption key release systems and residual losses to paying users are minimal.

In general, one can vary over the $(R, \delta)$ parameter space to map to hacker motivations and then gain insights into the equilibria that unfold when ransomware has characteristics consistent with each motivation. This parametric approach is preferable here because the wide-ranging and disparate motivations behind observed ransomware would make objective specification (in malicious agent modeling) untenable. Moreover, it permits broader insights into a threat landscape that is quite dynamic in nature; the version and intent of ransomware seen recently in WannaCry and NotPetya may look starkly different than the successful ransomware campaign of tomorrow which our model also intends to inform upon.

The consumer action space is $S = \{(B, P), (B, NP, R), (B, NP, NR), (NB)\}$ and for a given strategy profile $\sigma : V \to S$, the expected utility function for consumer $v$ is given by:

$$U_{RW}(v, \sigma) \triangleq \begin{cases} 
  v - p - c_p & \text{if } \sigma(v) = (B, P) ; \\
  v - p - \pi_r u(\sigma)(R + \delta \alpha v) & \text{if } \sigma(v) = (B, NP, R) ; \\
  v - p - \pi_r u(\sigma)\alpha v & \text{if } \sigma(v) = (B, NP, NR) ; \\
  0 & \text{if } \sigma(v) = (NB) ,
\end{cases}$$

(1)

where $u(\sigma) \triangleq \int_V \mathbb{1}_{\{\sigma(v) \in \{(B, NP, R), (B, NP, NR)\}\}} dv$ is the size of the unpatched adopting population in the presence of the ransomware threat. Without loss of generality, we assume that

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8See Section 6 where we discuss the limitations.
\[ \delta \in [0, 1], \pi_r \in (0, 1), c_p \in (0, 1), R \in [0, \infty), \text{and } \alpha \in (0, \infty). \]

### 3.2 Consumer Market Equilibrium

Before examining the impact of ransomware on the vendor’s decision, we first must characterize how consumers behave in equilibrium for a given price. There are two factors that complicate their decisions. First, the level of risk upon being unpatched is endogenously determined by the actions of consumers. Second, this risk includes the behavior of both those who would pay ransom as well as those who would not. Thus, we first focus on understanding the effect of their strategic interactions on equilibrium behavior due to the externality generated by both subpopulations. The consumer with valuation \( v \) selects an action that solves the following maximization problem: 

\[
\max_{s \in S} U_{RW}(v, \sigma),
\]

where the strategy profile \( \sigma \) is composed of \( \sigma_{-v} \) (which is taken as fixed) and the choice being made, i.e., \( \sigma(v) = s \). We denote the optimal action that solves her problem with \( s^*(v) \). Further, we denote the equilibrium strategy profile with \( \sigma^* \), and it satisfies the requirement that \( \sigma^*(v) = s^*(v) \) for all \( v \in \mathcal{V} \). We next characterize the structure of the consumer market equilibrium that arises.

**Lemma 1.** Given a price \( p \) and a set of parameters \( \pi_r, \alpha, c_p, R, \) and \( \delta \), there exists a unique equilibrium consumer strategy profile \( \sigma^* \) that is characterized by thresholds \( v_{nr}, v_r, v_p \in [0, 1] \).

For each \( v \in \mathcal{V} \), it satisfies

\[
\sigma^*(v) = \begin{cases} 
(B, P) & \text{if } v_p < v \leq 1; \\
(B, NP, R) & \text{if } v_r < v \leq v_p; \\
(B, NP, NR) & \text{if } v_{nr} < v \leq v_r; \\
(NB) & \text{if } 0 \leq v \leq v_{nr}.
\end{cases}
\]  

(2)

Lemma 1 establishes that the consumer market equilibrium has a threshold structure. The highest-valuation consumers have the most value to lose if attacked, so they patch in equilibrium (if the risk is sufficiently high). Those with lower valuations remain unpatched, trading off a fixed cost of patching for valuation-dependent losses. Of those who are unpatched, those with higher valuations are the ones who pay ransom to reduce the impact of being unpatched on their valuation-dependent losses. Importantly, it can be the case that no unpatched consumer (if \( R \) or \( \delta \) is sufficiently high) pays ransom or even all unpatched consumers pay ransom (if \( R \) or \( \delta \) is sufficiently low).

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\(^9\)For consumers in our model, at the point of decision making on whether to pay ransom they are only trading off ransom and residual losses, \( R + \delta \alpha v \), and full valuation-dependent losses, \( \alpha v \); all other costs are sunk at that point in time. In that \( \alpha \) can be greater than 1 when the attacks greatly affect firms, \( R \) can analogously be larger than 1 in situations where attackers are capitalizing on the ex-post value consumers are now concerned with mitigating.
We denote the vendor’s profit function by \( \Pi(p) = p \int I_{\{\sigma^*(v|p) \in \{(B,NP,NR),(B,NP,R),(B,P)\}\}} dv \), noting that marginal costs are assumed to be negligible for information goods. The vendor sets a price \( p \) for the software by solving the following problem: \( \max_{p \in [0, \infty)} \Pi(p), \) such that \((v_{nr}, v_r, v_p)\) are given by \( \sigma^*(\cdot | p) \). With the optimal price \( p^* \) that solves the vendor’s problem, we denote the associated profits by \( \Pi^* \triangleq \Pi(p^*) \). In the next section, we will discuss how \( R \) and \( \delta \) play a role in impacting the vendor’s pricing strategy and, ultimately, in shaping equilibrium consumer behavior.

3.3 Impact of Ransomware Characteristics

3.3.1 Equilibrium Market Structure

Because our model parameter space induces many different equilibria including those not commonly found in practical settings, it is worthwhile to focus our analysis on subspaces that are more relevant. As a simple example, it is natural that, in equilibrium, if patching costs \( (c_p) \) are too high, then no user patches; however, this outcome is not characteristic of settings that are commonly observed. To better focus on regions where key trade-offs are more active, we make the following assumptions going forward:

**Assumption 1.** \( 0 < c_p < 2 - \sqrt{3} \), and

**Assumption 2.** \( \frac{2}{(1-c_p)^2} - 2 < \alpha < 2(2 - c_p)^2 \).

While costs of patching involve inefficiencies due to downtime during the patching process, usually the patch distribution and installation processes have been greatly streamlined and sometimes automated for individual consumer and enterprise systems. When patching is done properly and in tandem with the adoption of fail-safe measures (such as restoration capabilities / backups) the associated business costs are usually within reasonable ranges. Similarly, such fail-safe measures can reduce the extent of damage of a ransomware (or other type of malware) attack. For simplicity, in our model, we assume that patching is effective at preventing the exploitation of the vulnerability. Moreover, the assumptions on \( c_p \) and \( \alpha \) above are sufficient conditions to obtain the findings in our paper, which can extend well beyond this focal region.

It will be helpful to better understand how the vendor changes the consumer market structure he induces via pricing in equilibrium based on some characteristics of the ransomware setting, e.g. size of the ransom demand \( R \) and security risk factor level \( \pi_r \). Figure 1 provides a helpful illustration of how the consumer market equilibrium outcome fluctuates. This figure depicts an instance of the focal region we study (defined by the assumptions
Figure 1: Characterization of equilibrium consumer market structures across regions in the ransom demanded ($R$) and security loss factor ($\pi_r$). Region labels describe the consumer segments that arise in each region in order of increasing consumer valuations (from left to right). Patching costs ($c_p = 0.12$), security loss factor ($\alpha = 0.8$), and residual loss factor ($\delta = 0.05$) are selected to ensure all consumer patching and ransom paying behaviors are present for some sub-region.

Moreover, some subsequent figures study vertical and horizontal slices across Figure 1 which can be more easily visualized here. For consistency, the capital letter labels in these cases also refer back to the region labels of Figure 1.

Figure 1 shows that when $R$ and $\pi_r$ are sufficiently low, then prices are set such that all customers of the software opt to remain unpatched and pay the ransom if hit. Overall, the expected losses are sufficiently low in Region (A) that consumers do not find it worthwhile to incur the cost to protect themselves by patching, and, if hit, they also prefer to pay the ransom because $R$ is relatively low. On the other extreme, if $R$ and $\pi_r$ are sufficiently high, 

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Consumer Market Segments Represented

<table>
<thead>
<tr>
<th>Region</th>
<th>Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>Not Using / Paying Ransom</td>
</tr>
<tr>
<td>(B)</td>
<td>Not Using / Paying Ransom / Patching</td>
</tr>
<tr>
<td>(C)</td>
<td>Not Using / Not Paying Ransom / Paying Ransom / Patching</td>
</tr>
<tr>
<td>(D)</td>
<td>Not Using / Not Paying Ransom / Patching</td>
</tr>
<tr>
<td>(E)</td>
<td>Not Using / Not Paying Ransom / Paying Ransom</td>
</tr>
<tr>
<td>(F)</td>
<td>Not Using / Not Paying Ransom</td>
</tr>
</tbody>
</table>

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above), and it will be useful as a reference for the reader to keep in mind which structures are in play. Moreover, some subsequent figures study vertical and horizontal slices across Figure 1 which can be more easily visualized here. For consistency, the capital letter labels in these cases also refer back to the region labels of Figure 1.
then the equilibrium outcome would be identical to the outcome observed in a world without an opportunity to pay ransom to mitigate losses. This is seen in Region (D), in which the equilibrium outcome is $0 < v_{nr} < v_p < 1$. In a setting with a high security risk factor $\pi_r$, higher-valuation consumers have a strong incentive to protect themselves from risk by patching. Those with lower valuations prefer to remain unpatched and do not pay high ransom demands; this results in the market outcome described. In the middle ground between these two scenarios, we see that the equilibrium outcome aligns well with observations of the world today. In Region (C), the consumer market outcome is characterized by $0 < v_{nr} < v_r < v_p < 1$, in which some of the customers who opt to remain unpatched choose to pay the ransom if hit. If the risk factor is not as high as in Region (C), then one might expect $0 < v_{nr} < v_r < 1$ to arise in equilibrium, in which no customer patches and higher-valuation customers opt to pay ransom if hit. Such a region does arise, and it is depicted as Region (E). Similarly, with a high risk factor but a smaller expected ransom demand, the outcome $0 < v_r < v_p < 1$ in which the highest-valuation customers patch and all unpatched customers pay the ransom if hit also arises, depicted as Region (B). Lastly, when the risk factor is low but the ransom amount is high, consumers do not have an incentive to patch (due to the low risk) and, at the same time, if struck by an attack, they will not pay the high ransom, inducing an equilibrium $0 < v_{nr} < 1$, as seen in Region (F).

### 3.3.2 Role of the Ransom Amount, Risk and Residual Losses

With the newfound understanding of how the equilibrium outcome unfolds across different regions of the parameter space, we next investigate regions of interest in more depth. In the rest of this section, we describe and illustrate several insights into ransomware economics. For example, one might expect that a higher ransom demand would negatively impact the vendor and reduce the market share of the affected product. However, that is not always the case, as is shown in the first proposition. For the majority of results and discussions in this paper, we focus on residual losses for ransom-paying consumers being reasonably low (i.e., $\delta$ satisfying an upper bound) such that paying ransom can be incentive compatible. This assumption matches more recent ransomware trends (Disparte 2018). An economically-motivated hacker would generally deploy ransomware with characteristics satisfying such conditions because the hacker’s goal is to generate ransom payments which would be negatively impacted by post-payment malicious behavior. Despite ransom-paying consumers requiring some belief about “honor among thieves”, for certain classes of hacker motivations maintaining this honor would be in everyone’s best interest (Fleishman 2016). To gain a broader view into diverse motivations and provide an overall more complete analysis, we relax this assumption.
in Proposition 5 and again in Section 5.1 to discuss scenarios of intermediate and high residual losses.

**Proposition 1.** There exists bound $\tilde{\delta} > 0$ such that if $\delta < \tilde{\delta}$ and $\pi_r > \bar{\pi}_r$, then:

(a) if $0 \leq R < R_1$, then the equilibrium consumer market structure is $0 < v_r < 1$. As $R$ increases, so does the vendor’s price but market size and profits decrease;

(b) if $R_1 \leq R < R_2$, then the equilibrium consumer market structure is $0 < v_r < v_p < 1$. As $R$ increases, the vendor’s price, market size, and profits all decrease;

(c) if $R_2 \leq R < R_3$, then the equilibrium consumer market structure is $0 < v_{nr} < v_r < v_p < 1$. As $R$ increases, the vendor’s price, market size, and profits all increase;

(d) if $R \geq R_3$, then the equilibrium consumer market structure is $0 < v_{nr} < v_p < 1$, and there exists $\omega > R_3$ such that, as $R$ increases,

(i) the vendor’s price and profits increase while the market size decreases on $R < \omega$;

(ii) the vendor’s price, market size, and profits are constant on $R \geq \omega$.\(^{10}\)

Proposition 1 is illustrated in Figure 2 which depicts how the consumer market structure, vendor price and profit react to changes in the ransom amount, $R$.\(^{11}\) For the parameters used in Figure 2, the thresholds identified in Proposition 1 are computed to be approximately $R_1 = 0.34$, $R_2 = 0.41$, $R_3 = 0.58$, and $\omega = 0.59$. When the ransom demand is not too low and the potential losses from the attack are sufficiently high, then high-value consumers elect to patch ex ante. This patching behavior can be seen entering into panel (a) of Figure 2 for $R \geq 0.34$. It is important to note that the trade-off here centers on $c_p$ versus $\pi_r u(\sigma)(R + \delta \alpha v)$ (i.e., the expected costs under a ransom-paying strategy). Therefore, a patching population only emerges when $R$ is large relative to $c_p$, because the likelihood of an attack striking, $\pi_r u(\sigma)$, can be low in equilibrium. Against this backdrop, part (a) of Proposition 1 establishes that a patching strategy does not emerge for sufficiently small ransom demands. There are only unpatched consumers who all pay ransom if an attack arises, as is depicted in panel (a) of Figure 2 for $R < 0.34$; this is also identified as falling under Region (A) of Figure 1. Hence, in equilibrium, all purchasing consumers are directly and negatively impacted by a higher

\(^{10}\)The existence, characterization, and relative ordering (e.g., $0 < R_1 < R_2 < R_3$) of the presented bounds are formally established in the proof under the focal region (see Assumptions 1 and 2), noting that $R_1 \rightarrow \frac{(2-c_p)c_p}{1-c_p},$ $R_2 \rightarrow \frac{\alpha}{c_p},$ $R_3 \rightarrow \frac{\alpha \sqrt{\pi_r} + \sqrt{\alpha (16c_p + \alpha \pi_r)}}{4c_p \sqrt{\pi_r}},$ and $\bar{\pi}_r \rightarrow \frac{c_p(2-c_p)^2}{\alpha (1-c_p)}$ as $\delta$ becomes small.

\(^{11}\)The impact to the consumer market structure itself can also be viewed as a cross-section of Figure 1 horizontally at $\pi_r = 0.75.$
Figure 2: Impact of ransom demand ($R$) on the equilibrium market outcome, vendor’s price, and the vendor’s profit. The parameter values are $c_p = 0.12$, $\alpha = 0.8$, $\delta = 0.05$, and $\pi_r = 0.75$. The capitalized letter region labels correspond to region labels in Figure 1. The legend in panel (b) also applies to panel (c).

$R$. In this situation, the vendor elevates price in order to reduce the size of the unpatched population and help mitigate the risk of an attack. Throughout this region, increases in $R$ will hurt vendor profits.

Part (b) of Proposition 1 pertains to a region of the parameter space in which the ransom demand is still low enough that all unpatched consumers simply pay the ransom if hit (but not so low that nobody patches). This can be observed for $0.34 \leq R < 0.41$ in Figure 2, which falls under Region (B) of Figure 1. Similar to region (A), unpatched consumers are again negatively impacted by an increase in $R$. Higher valuation consumers now patch, which provides risk relief to unpatched consumers. The vendor has less incentive to use a high price to contain risk via a reduced user population, therefore as $R$ moves from Region (A) to Region (B), he chooses to drop price and expand the market. Nevertheless, within Region (B), as $R$ increases further, to reduce the additional burden on lower valuation consumers he gradually decreases price. The vendor does so in a controlled way, without completely compensating for the increase in expected losses, thus leading to a gradual shrinking of the
market. In that way, a decreasing price and market size ultimately hurt profitability as is depicted in panel (c) of Figure 2.

As $R$ moves relatively higher, all three adopter segments emerge, with the unpatched population splitting into subpopulations of ransom payers and non-payers, while high valuation consumers elect to remain patched. This is captured in Figure 2 from $0.41 \leq R < 0.58$ and corresponds to Region (C) of Figure 1. An increase in $R$ incentivizes some unpatched consumers who would have paid ransom to strictly prefer patching over risking being hit with ransomware. On the other hand, unpatched consumers with lower valuations are not directly impacted by an increase in $R$ because they do not pay ransom anyway. However, these consumers are still indirectly impacted by $R$ because the negative externality that they endure upon remaining unpatched is reduced by the increased patching behavior of high valuation consumers. As a result, all else being equal, these low valuation unpatched consumers are now better off under a higher ransom demand being charged. The market size therefore expands in $R$, as non-adopters now find it beneficial to adopt the product. In turn, the vendor is able to profitably extract additional surplus by charging a higher price. This outcome is described in part (c) of Proposition 1.

When the ransom demand is sufficiently high (i.e., $R \geq 0.58$ in Figure 2 and corresponding to Region (D) of Figure 1), then no unpatched customer pays ransom if hit. When the ransom level hits the terminal level (i.e., $R \geq 0.59$ in Figure 2), it becomes naturally cost prohibitive from the consumer perspective, hence market size and vendor price and profit do not change in $R$. However, an interesting outcome occurs in the range in between, corresponding to $R < \omega$ in part (d.i) of Proposition 1. Specifically, price and vendor profit are increasing in $R$ despite the absence of ransom-paying consumers. To understand why, note that in Region (C) as $R$ increases, the vendor uses price to throttle demand in such a way that ransom-paying consumers are gradually removed as a segment. Because paying the ransom is more feasible in Region (C), the vendor is constrained to not charge a very high price. If he does so, low valuation consumers would be pushed out, reducing the market size which reduces overall risk. But, in turn, this creates incentives for higher valuation consumers to again pay ransom instead of patching, which increases the risk right back. The net impact of these effects is that the overall market size shrinks too much to make a high price strategy effective in Region (C).

As $R$ increases at the beginning of Region (D), the ransom paying option is now even worse for any existing consumer and there are already no ransom-paying consumers in equilibrium. Yet, very close to the transition point, the ransom-paying option is still viable for consumers over the set of reasonable prices the vendor might choose. Setting the price immediately to the higher equilibrium level observed for $R \geq \omega$ is not optimal because the
ransom level is not sufficiently high yet; such a price would instead encourage some would-be patching consumers to pay ransom, leading to a suboptimal risk level in the market. Nevertheless, as \( R \) increases, the vendor can also gradually increase price as the concern of having some consumers choosing to pay ransom subsides. As it turns out, the vendor will optimally price in such a way that the boundary between patched and unpatched segments is actually a triple indifference point characterizing consumers that are indifferent between patching, not patching and paying ransom, and not patching and not paying ransom. In that sense, for ransom amounts satisfying \( R < \omega \) in this region, the vendor prices at the highest point that just prevents consumers from paying ransom. Surprisingly, in this region, the vendor’s profit also increases in \( R \) because the price hike compensates for the market size loss.

By offering victims a chance to reduce their losses, ransomware attackers can sometimes segment the unpatched user population into two interdependent tiers. The expansion or reduction of either tier indirectly impacts both tiers simultaneously because all unpatched hosts are potential vectors for the spread of ransomware. But now, in contrast to traditional modes of attack, an increase in the ransom demand may directly affect only a single tier which helps the vendor to discriminate. As we will see, because of these characteristics, ransomware also gives rise to some unique pricing strategies.

Besides understanding how the market is expected to evolve as attackers demand higher ransoms, we also want to explore how the market changes as the inherent risk (\( \pi_r \)) of the software being breached increases. We first analyze how the vendor adjusts his pricing strategy with respect to risk.

**Proposition 2.** There exist bounds \( \tilde{\delta} > 0 \) and \( \hat{\omega} > \frac{\alpha}{2-c_p} \) such that if \( \delta < \tilde{\delta} \), then:

(a) if \( 0 \leq R \leq \hat{R}_1 \), then the vendor’s equilibrium price is continuously decreasing in \( \pi_r \);

(b) if \( \hat{R}_1 < R \leq \hat{R}_2 \), then the vendor’s equilibrium price is piecewise decreasing in \( \pi_r \) on adjacent intervals \((0, \hat{\pi})\) and \((\hat{\pi}, 1)\) while jumping downward at \( \hat{\pi} \);

(c) if \( \hat{R}_2 < R \leq \hat{\omega} \), then there exists \( \hat{\pi} \) such that the vendor’s equilibrium price is piecewise decreasing in \( \pi_r \) on adjacent intervals \((0, \hat{\pi})\), \((\hat{\pi}, \bar{\pi})\), and \((\bar{\pi}, 1)\). His strategy is discontinuous in \( \pi_r \): the price should be jumped up at \( \hat{\pi} \) and significantly jumped down at \( \bar{\pi}_1 \).\(^{12}\)

\(^{12}\)The existence, characterization, and relative ordering of the presented bounds are formally established in the proofs under the focal region, noting that \( \hat{R}_1 \to \frac{1}{(1-c_p)^2} - 1 \), \( \hat{R}_2 \to \frac{\alpha}{2} \), \( \pi_1 \to \frac{(2-c_p)c_p}{(1-c_p)^2} R \), and \( \pi_2 \to \frac{c_p}{\alpha} R \) as \( \delta \) becomes small, and \( \bar{\pi} = \min(\pi_1, \pi_2) \). Implicit bounds \( \hat{\omega} \) and \( \hat{\pi} \) are characterized as such in Lemma A.7 of Section A.3 of the Appendix.
When the level of ransom demand is very low \((R \leq \hat{R}_1)\), all consumers prefer to remain unpatched and pay ransom if hit. Therefore, as risk increases, all purchasing consumers are directly affected by the associated increase in expected ransom payments. In order to throttle consumers from discontinuing use while also ensuring the risk level stays in check, the vendor gradually decreases price in a controlled way.

At slightly higher ransom levels \((\hat{R}_1 < R \leq \hat{R}_2)\), for low enough risk, the behavior is the same as is described above. However, once risk passes a certain threshold, \(\pi_1\), high valuation consumers now find it incentive compatible to patch, leading to a consumer market characterized by \(0 < v_r < v_p < 1\). This change in behavior gives rise to potential relief in terms of overall risk. The vendor might seek to extract associated surplus, but this is not a profitable strategy. Rather, as risk transitions into this region, the vendor discontinuously drops its price to expand market coverage at the lower end. From there, the vendor manages further increases in risk through gradual downward price adjustments. This outcome is described in part (b) of Proposition 2.

Once ransom demands increase to a range characterized by richer trade-offs \((\hat{R}_2 \leq R \leq \hat{\omega})\), a more complex pricing strategy unfolds. We formally describe this strategy in part (c) of Proposition 2. Figure 3 illustrates this particular scenario, with panel (b) explicitly capturing the price sensitivity with respect to risk. The corresponding cutoff points are \(\hat{\pi} \approx 0.424\) and \(\tilde{\pi} \approx 0.568\). When the inherent risk factor is low, i.e., \(\pi_r < \hat{\pi}\), consumers do not patch. However, ransom demands are higher in this region, inducing consumers to segment in their decision to pay ransom. In particular, the unpatched population separates into two subpopulations in terms of their equilibrium strategies (those who pay and those who do not pay ransom if hit) leading to a consumer market outcome characterized by \(0 < v_{nr} < v_r < 1\). This outcome corresponds to Region (E) in panel (a) of Figure 3. As the inherent risk factor increases through this range, the vendor mitigates the impact of increased risk on his customers by lowering price.

However, when the risk factor increases further and crosses the threshold at \(\hat{\pi}\), there is a significant and strategic change in the equilibrium pricing behavior of the vendor. This is the primary message contained in part (c) of Proposition 2: when the risk factor lies in a middle range (i.e., \(\pi_r \in (\hat{\pi}, \tilde{\pi})\)), the vendor implements a strategically higher price to focus only on higher-valuation customers. This pricing strategy changes the consumer market characterization to \(0 < v_r < 1\) in which all purchasing consumers remain unpatched and pay ransom if hit, illustrated by Region (A) in panel (a) of Figure 3. Lower-valuation customers drop out of the market due to the increased risk and higher price. Notably, as we will see later in Section 5.1, such a strategic discontinuous increase in pricing is not observed in a comparative setting where ransomware is not present in the market.
Figure 3: Impact of the risk factor ($\pi_r$) on the equilibrium market outcome, vendor’s price, size of the market segment willing to pay ransom if hit, endogenous risk level, and expected total ransom paid. The parameter values are $c_p = 0.12$, $\alpha = 0.8$, $\delta = 0.05$, and $R = 0.43$.

As the inherent risk factor moves up higher, exceeding $\tilde{\pi}$, consumers naturally have a much higher incentive to patch their systems as a means to protect themselves. Because they would already need to incur the patching cost of $c_p$, the employment of a high price strategy becomes incompatible because it now overly restricts usage. Instead, once the inherent risk factor increases to the threshold $\tilde{\pi}$, the vendor again adapts his strategy to facilitate consumer patching, reduce the security externality, and substantially increase usage of his software. He achieves this via a discontinuously lower price at $\tilde{\pi}$, with gradual price reductions as risk increases thereafter. There are two market structures that potentially emerge under high risk, depending on the ransom amount: either $0 < v_r < v_p < 1$ (if $R$ is at the lower end of $[\hat{R}_2, \hat{\omega}]$) or $0 < v_{nr} < v_r < v_p < 1$ (if $R$ is at the higher end of the same interval).\textsuperscript{13} In Figure 3, we illustrate the latter, in which a large ransom incentivizes a segment of lower valuation consumers to forgo paying ransom while remaining unpatched.

\textsuperscript{13}For details on these two consumer market structures that arise, see parts (c) and (d) of Lemma A.8 of Section A.3 of the Appendix.
When examining the impact of the inherent risk factor on the software market, one question of interest is how the expected total ransom paid by victims is affected. Although the size of the consumer population willing to pay ransom if hit (r) always shrinks as the risk factor increases (as seen in panel (c) of Figure 3), the expected total ransom paid is both non-monotone and exhibits greater complexity (as seen in panel (e)).

**Proposition 3.** Under the conditions of Proposition 2,

(a) the vendor’s profit and market size are piecewise decreasing in \( \pi_r \);

(b) if \( 0 < R < R_2 \), then the expected total ransom paid is piecewise increasing in \( \pi_r \). Moreover, the size of the ransom-paying population decreases in \( \pi_r \) if and only if \( \pi_r \in (\hat{\pi}, \pi_1) \);

(c) if \( R_2 \leq R < \hat{\omega} \), the expected total ransom paid is piecewise increasing in \( \pi_r \) on \((0, \pi_2)\) and decreasing in \( \pi_r \) on \((\pi_2, 1)\). The size of the ransom-paying population piecewise weakly decreases in \( \pi_r \) everywhere.\(^{14}\)

In part (a) of Proposition 3, the vendor’s profit and market size being piecewise decreasing in \( \pi_r \) is natural because every unpatched segment (whether paying ransom or not) is directly affected by the increased prospect of an attack. In that overall risk is endogenous, there will always be a segment of unpatched consumers in the market who are sensitive to ransomware attacks.

The more material takeaway lies in parts (b) and (c) of Proposition 3 where we establish there exists a risk region in which the total expected ransom paid increases in \( \pi_r \) while the ransom paying population size decreases. We focus our discussion on the scenario presented in part (c), which is the richer of the two due to the associated market outcome and is also the scenario depicted in Figure 3. In particular, panel (c) plots the size of the consumer segment that pays ransom if hit, \( r(\sigma^*) \), i.e., the mass of consumers whose equilibrium strategy is \((B, NP, R)\). Panel (d) illustrates the endogenous risk level, \( \pi_r u(\sigma^*) \), where \( u(\sigma^*) \) measures the total mass of consumers who remain unpatched in equilibrium, choosing either \((B, NP, NR)\) or \((B, NP, R)\). Panel (e) illustrates the expected total ransom paid, i.e., \( T(\sigma^*) \equiv \pi_r u(\sigma^*) r(\sigma^*) R \). The key point to note here is that the externality \( u(\sigma^*) \) depends not only on those consumers willing to pay the ransom but also on those who remain unpatched and are not willing to pay. Referencing the left-hand side of panel (a) in Figure 3, i.e., Region (E), since the ransom demand \( R \) is moderate, the unpatched population splits into the two tiers when the inherent risk \( \pi_r \) is low (resulting in a market structure \( 0 < v_{nr} < v_r < 1 \)).

\(^{14}\)See footnote 12 for further specification of the bounds. A more comprehensive statement of Proposition 3 can be found in Section A.5 of the Appendix.
valuation of the indifferent consumer, between paying and not paying ransom \( v = \frac{R}{\alpha(1-\delta)} \), is independent of the risk factor because the tradeoff is only between incurring a loss of \( \alpha v \) and incurring a loss of \( R + \delta \alpha v \). Since consumers who remain unpatched but do not pay ransom have valuations even lower than this threshold, they are the ones to first drop out of the market as \( \pi_r \) increases. Consequently, the size of the consumer population willing to pay ransom if hit remains constant in \( \pi_r \) at first (see Region (E) of panel (c)). Although \( u(\sigma^*) \) necessarily shrinks, the overall risk \( \pi_r u(\sigma^*) \) increases as \( \pi_r \) increases (same region of panel (d)) such that the expected total paid by victims is also increasing in \( \pi_r \) (same region of panel (e)).

When the vendor strategically increases price (at the boundary between Regions (E) and (A) in Figure 3), low-valuation consumers who had been choosing \((B, NR, NP)\) drop out of the market. Additionally, the size of the ransom-paying group shrinks which now becomes the only segment present in the market (i.e., the market structure is given by \( 0 < v_r < 1 \)). As the market structure changes, the overall risk externality \( \pi_r u(\sigma^*) = \pi_r r(\sigma^*) \) also suddenly drops. The net impact of these two effects is a drop in the size of the expected total ransom paid by victims, which can be seen at \( \pi_r = \hat{\pi} \) in Figure 3. Nevertheless, as risk further increases within this range of the risk factor (i.e., \( \pi_r \in (\hat{\pi}, \pi_2) \)), the ransom-paying population does not decrease steeply which is depicted in Region (A) of panel (c). Hence, the expected total ransom paid by victims remains monotonically increasing in \( \pi_r \), albeit initially trailing behind the levels just prior to the change in market structure.

At the junction between Regions (A) and (C), as described in part (c) of Proposition 2, the vendor finds it optimal to significantly drop price so that all three market segments emerge in equilibrium (i.e., \( 0 < v_{nr} < v_r < v_p < 1 \)). This shift in pricing strategy invites additional consumers to enter at the low end of the market, who naturally remain unpatched. The increased risk they create induces high valuation consumers to shield themselves from risk by patching. In aggregate, the ransom-paying population \( r(\sigma^*) \) shrinks because more consumers switch from paying ransom to patching than from not using to paying ransom. The sudden jump in the unpatched population \( u(\sigma^*) \) compensates for the drop in \( r(\sigma^*) \), and overall the total expected ransom paid momentarily jumps upward. What is more notable is the change in monotonicity. As the risk increases even further, higher-valuation consumers who were willing to just pay ransom if hit switch to patching at a much steeper rate such that those low-valuation consumers who do not pay ransom are only marginally impacted by the increased risk. Moreover, as discussed before, the marginal customer indifferent between paying and not paying ransom is not affected by the overall risk level. Thus, both the ransom-paying population \( r(\sigma^*) \) and the overall unpatched population \( u(\sigma^*) \) keep shrinking. Nevertheless, unlike in regions of lower risk (i.e., Regions (E) and (A) in Figure 3), this dual shrinking
effect dominates the increase in the risk factor \((\pi_r)\) and the expected overall ransom paid decreases.

While the option to pay ransom offers a recourse to mitigate value-dependent losses, it also involves a secondary risk. When considering the ransom payment, victims in general are not sure a priori that the attacker will deliver the promised decryption keys. As mentioned in Section 3, this secondary risk is captured by the parameter \(\delta\). In the remainder of this section, we explore how \(\delta\) impacts the vendor’s profit, expected aggregate losses incurred by the unpatched population, and aggregate consumer surplus. The latter two measures are defined by:

\[
UL \triangleq \int_V \mathbb{1}_{\{\sigma^*(v) = (B,NP,NR)\}} \pi_r u(\sigma^*) \alpha v dv + \int_V \mathbb{1}_{\{\sigma^*(v) = (B,NP,R)\}} \pi_r u(\sigma^*)(R + \delta \alpha v) dv,
\]

\[
CS \triangleq \int_V \mathbb{1}_{\{\sigma^*(v) \in \{(B,NP,NR),(B,NP,R),(B,P)\}\}} U_{RW}(v, \sigma^*) dv.
\]

**Proposition 4.** There exists a bound \(\tilde{\delta} > 0\) such that if \(\delta < \tilde{\delta}\), \(R \in (R_2, R_3)\) and \(\pi_r > \bar{\pi}_r\) are satisfied, then:

(a) the vendor’s profit is increasing in \(\delta\);

(b) the market size, aggregate unpatched losses, and consumer surplus are all decreasing in \(\delta\).

Proposition 4 characterizes a market scenario that falls within Region (C) of Figure 1, in which case \(0 < v_{nr} < v_r < v_p < 1\) is the ensuing equilibrium outcome. The results in Proposition 4 can be observed in the range \(0 < \delta < 0.301\) in Figure 4. The vendor benefits from an increase in residual loss factor \(\delta\) for the same reason for which he benefits when \(R\) increases (as discussed in part (c) of Proposition 1). An increase in \(\delta\) only directly impacts ransom-paying unpatched consumers, providing a disincentive for them to adopt this strategy. As some consumers who were paying ransom switch to patching due to an increased risk of not receiving working decryption keys, the aggregate risk externality decreases and the vendor can increase his price while keeping the overall market relatively steady, extracting additional surplus (as seen in Figure 4). When all market segments are present in equilibrium (depicted in the left-hand portion of the panels in Figure 4), the vendor clearly prefers greater potential residual losses (whether stemming from failures with payment systems or decryption keys as well as mixed motivations of hackers) because it presents an unusual and counter-intuitive

\[\text{15The bounds } \bar{\pi}_r, R_2, \text{ and } R_3 \text{ are the same as in Proposition 1.}\]
opportunity to charge a premium for higher residual risk in the market without losing too many consumers.

Furthermore, in this residual loss range, the vendor would prefer that consumers have the worst possible perception regarding the trustworthiness of the attacker whereas an economically-driven hacker would prefer the opposite. Reports suggest that, compared to the early days of ransomware attacks, the market for such attacks has become efficient and the success rate in retrieving access to compromised assets following a ransom payment increased dramatically, highlighting prevalent economic motivations on the attacker side (Disparte 2018). But, in many cases, corporate victims that pay ransom do not publicize their actions (Cimpanu 2017), which makes it easier for the vendor to vilify attackers in an amplified way even when decryption keys are often returned. Even a small number of failed interactions can damage the hacker’s reputation and effectively cut off its revenue stream.

Proposition 4 further shows that the vendor’s expected profit can be the lowest at the same \( \delta \) that concomitantly gives the worst expected losses to the unpatched population.
and the highest overall consumer surplus, as seen in panel (d) of Figure 4. As $\delta$ increases, the ransom-paying unpatched population shrinks as customers at both ends of this segment choose different strategies (higher-valuation customers choose to patch, while lower-valuation customers choose not to pay ransom). Moreover, the overall unpatched population shrinks as well, thus lowering the security risk externality. The redistribution of consumers among segments and the reduced overall risk result in the aggregate expected losses to the unpatched population to be decreasing in $\delta$. Even though these unpatched losses are decreasing, the vendor employs a higher price to further throttle the population size and help mitigate the increased magnitude of residual losses. Given the relatively stable (but slightly shrinking) size of the market when $\delta < 0.301$, the reduction in losses to the unpatched population is dominated by the larger premium, hence consumer surplus also decreases in $\delta$.

Having fully explored the case where residual losses for ransom-paying consumers are low, we next turn our attention to the case where residual losses become high. An economically-motivated hacker may aptly be characterized as having a lower $\delta$ because revenue generation requires a mass of consumers to pay ransom in equilibrium. In particular, a lower $\delta$ helps to make this strategy incentive compatible for some subset of the consumer space. On the other hand, a politically-motivated hacker or a hacker otherwise motivated may be significantly less concerned about capping residual losses. In that generating ransom payments is no longer a primary concern, the practical range of $\delta$ for hackers with these motivations may become much broader. In this sense, it is worthwhile to also examine equilibrium outcomes when $\delta$ is high and explore how the relevant comparative statics are impacted.

**Proposition 5.** When the residual loss factor is high, satisfying $\delta > 1 - \frac{R}{\alpha}$, the vendor’s equilibrium price, the size of the market, and equilibrium profit are constant in $R$ and $\delta$. The consumer market outcome is given by $0 < v_{nr} < 1$ when patching costs are high and $0 < v_{nr} < v_p < 1$ when patching costs are low.\(^{16}\)

Proposition 5 formally demonstrates that as the residual losses become sufficiently high, consumers react in a predictable way in that it is no longer incentive compatible for any user to pay ransom in equilibrium. Rather than pay ransom, users will either all remain unpatched and risk losses if the cost of patching is prohibitive or split between staying unpatched and patching otherwise. Note that in the absence of a ransom-paying population, these are the only two possible consumer market structures that arise in equilibrium and, as illustrated above, each of these outcomes is possible depending on the parameter region. As can be seen in the right-hand side of all panels in Figure 4, the relevant measures all become constant

\(^{16}\)The terms “high” and “low” are used to describe relative position within the focal region as defined in Assumption 1.
in $\delta$ once the residual loss factor has become sufficiently high; for the particular parameter set, patching costs are low, leading to some users patching in equilibrium.

As described in Section 3.1, a higher $\delta$ can reflect a number of different scenarios including dishonest attackers, faulty encryption and communication, and political agendas. The outcome here naturally converges to that under a model without ransomware as we show in Section 5.1. One might then think of such a simpler model as one that captures the outcome under a politically-motivated hacker. However, in Section 5.1, we will discuss why this can be misleading despite the seeming equivalence between the two in the current cybersecurity landscape.

4 Other Ransomware Attack Vectors

In the prior section, we focused on threats on patchable vulnerabilities. Nevertheless, ransomware can also get a foothold inside networks via other infection vectors. Such examples include ransomware spread via zero-day vulnerabilities (for which there is no patch available at the time of the attack), social engineering campaigns (phishing) that lead to credential theft and malware installation by exploiting the human factor (Gendre 2019a; Goodin 2019). For example, one vector that attackers have used in recent years is Microsoft’s collaboration platform SharePoint (Guida 2018; Gatlan 2019; Gendre 2019b). Attackers send potential victims links to SharePoint documents, which upon being clicked lead these victims to spoofed Office 365 login pages. By stealing credentials, attackers can then send additional phishing emails or SharePoint documents with ransomware attached from within the victim’s organization to other organizations (such as suppliers or clients). Because SharePoint documents are passed, Microsoft cannot blacklist links to these documents without negatively impacting users’ ability to collaborate. Consequently, all users are faced with this risk, and as more customers use the service, the more attractive the attack vector becomes. We also discussed in the Introduction several other ransomware attacks on Microsoft Office and Oracle WebLogic server software that were not patchable. In all of these instances, only adopters of a particular software or service are vulnerable to such an attack, which means interdependent risk is being induced by the entire user population.$^{17}$

We construct a modified model to capture the unique aspects of other common ransomware attack vectors and refer to it as $RW-OV$ (ransomware, other vectors):

$^{17}$There are other types of unpatchable attacks that are not at an application level (e.g., phishing campaigns designed to syphon user credentials by rerouting users to a copy-cat web page replicating that of an official service provider). Such attacks are not captured through the lens of an economic decision to adopt a single software package in isolation, as they in general also impact non-users of that particular package.
$$U_{RW-OV}(v, \sigma) \triangleq \begin{cases} 
 v - p - \pi_r n(\sigma)(R + \delta \alpha v) & \text{if } \sigma(v) = (B, R); \\
 v - p - \pi_r n(\sigma) \alpha v & \text{if } \sigma(v) = (B, NR); \\
 0 & \text{if } \sigma(v) = (NB), 
\end{cases} \quad (3)$$

where $n(\sigma) \triangleq \int_{\mathcal{V}} \mathbb{I}_{\{\sigma(v) \in \{(B,R),(B, NR)\}\}} dv$ is the total size of the adopting population in the presence of this class of ransomware threat. Unlike in the case of an attack on a patchable vulnerability, all users are exposed to this risk; hence, in this case, risk interdependence is tied directly to usage.

One of the primary goals of this section is to make meaningful comparisons of outcomes under model $RW-OV$ with those established in the propositions in Section 3 for model $RW$. Analogous to Assumption 2 in Section 3, we similarly focus on a sub-range of the loss factor (specifically, $\alpha \in (\sqrt{3}, 6)$) which places the models on similar footing.\textsuperscript{18} The following lemma describes the consumer market equilibrium:

**Lemma 2.** Under model $RW-OV$, given a price $p$ and a set of parameters $\pi_r, \alpha, R$, and $\delta$, there exists a unique equilibrium consumer strategy profile $\sigma^*$ that is characterized by thresholds $v_{nr}$ and $v_r \in [0, 1]$. For each $v \in \mathcal{V}$, it satisfies:

$$\sigma^*(v) = \begin{cases} 
 (B, R) & \text{if } v_r < v \leq 1; \\
 (B, NR) & \text{if } v_{nr} < v \leq v_r; \\
 (NB) & \text{if } 0 \leq v \leq v_{nr}. 
\end{cases} \quad (4)$$

For this structure, some of the consumer segments may not appear as thresholds collapse into each other; however, if they appear, the segments will be ordered as presented in equation (4).

Analogous to Figure 1, we illustrate in Figure 5 the market equilibria (under equilibrium pricing), based on a wide range of combinations of $\pi_r$ and $R$. To be consistent with model $RW$, we use the same notation for regions - (A), (E), (F) - as in Figure 1. The market structures that emerge under model $RW-OV$ are relatively less complex and are formally presented in the following proposition.

**Proposition 6.** Under model $RW-OV$, there exist bounds $\tilde{\delta} > 0$ and $\tilde{R}_1 \leq \alpha(1 - \delta)$ such that if $\delta < \tilde{\delta}$, then:

\textsuperscript{18}In Assumption 2, the lower bound on $\alpha$ is increasing in $c_p$, while the upper bound on $\alpha$ is decreasing in $c_p$. Thus, taking the upper bound on $c_p$ in Assumption 1, i.e., $2 - \sqrt{3}$, and replacing in Assumption 2, we obtain the interval $(\sqrt{3}, 6)$ which is nested inside all other intervals $(\frac{2}{1-c_p} - 2, 2(2 - c_p)^2)$ for any $c_p$ satisfying Assumption 1. Therefore, for any such $c_p$, we can compare and contrast the outcomes under $RW$ and $RW-OV$ models directly provided that $\alpha \in (\sqrt{3}, 6)$. 

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Figure 5: Characterization of equilibrium consumer market structures across regions in the ransom demanded \( (R) \) and security loss factor \( (\pi_r) \). Region labels describe the consumer segments that arise in each region in order of increasing consumer valuations (from left to right). Security loss factor \( (\alpha = 0.8) \) and residual loss factor \( (\delta = 0.05) \) are selected to ensure all strategies are present for at least some sub-region.

(a) if \( R \leq \tilde{R}_1 \), then the equilibrium consumer market structure is \( 0 < v_r < 1 \). As \( R \) increases, so does the vendor’s price but market size and profits decrease;\(^{19}\)

(b) If \( R > \tilde{R}_1 \), then the equilibrium consumer market structure is \( 0 < v_{nr} < v_r < 1 \) when \( R < \alpha(1 - \delta) \) and \( 0 < v_{nr} < 1 \) when \( R \geq \alpha(1 - \delta) \). As \( R \) increases, the vendor’s price, market size, and profits are constant in \( R \).

In part (c) of Proposition 1 (depicted by Region (C) in Figure 2), we learned that with ransomware attacks on patchable vulnerabilities, it is possible for the vendor’s price, market size and profits to simultaneously increase in \( R \). Under model \( RW-OV \), such a comparative static can no longer arise, as we further explain below. This highlights the importance that the patching option has on the vendor’s strategy when facing a ransomware threat. Specifically, the market can expand and the vendor can be better off with increasing ransoms only in a context of ransomware attacks on patchable vulnerabilities.

In Figure 6, we illustrate how market structure, price, and vendor profit change in the ransom amount \( R \) under \( RW-OV \). For the parameter set employed in the figure, \( \tilde{R}_1 \) is approximately 0.47. In Region (A), when the ransom amount is low, all adopters would prefer to pay the ransom if successfully attacked. Within this region, as the ransom level increases, the firm controls risk in the market by maintaining a high price and actually gradually increasing it. This, together with the increasing ransom level, effectively reduce the market

\(^{19}\)The characterization of \( \tilde{R}_1 \) is provided in Lemma A.9 of Section A.3 of the Appendix.
Figure 6: Impact of ransom demand ($R$) on the equilibrium market outcome, vendor’s price, and profit. The parameter values are $\alpha = 0.8$, $\delta = 0.05$, and $\pi_r = 0.75$. The capitalized letter region labels correspond to region labels in Figure 5. The legend in panel (b) also applies to panel (c).

size, and hence interdependent risk. However, profits are declining in $R$ because the limited increase in price does not compensate for the large reduction in market size.

Beyond $\tilde{R}_1$, this strategy is no longer tenable. Instead, the firm lowers price in an effort to expand the market at the bottom and bring in consumers who balk at the size of the ransom request. The marginal adopter has a valuation satisfying the equation $v - \pi_r(1 - v)\alpha v = p$, which is no longer impacted by the ransom amount. As such, market size, price and profits are no longer sensitive in $R$ in Regions (E) and (F). As the ransom increases in Region (E), while the overall market size stays constant, the market structure continues to be characterized by a declining ransom paying segment. Once ransom values attain a high level (Region (F)), all adopters ignore the ransom demand and prefer to incur full security losses.
In the next two propositions, we establish that some primary comparative statics we characterize using our main model \((RW)\) are robust to the class of ransomware attacks being modeled under \(RW-OV\). First, we examine how the inherent risk factor impacts the vendor’s price for ransomware in the presence of alternative vectors to patchable vulnerabilities.

**Proposition 7.** Under model \(RW-OV\), there exists a bound \(\tilde{\delta} > 0\) such that if \(\delta < \tilde{\delta}\), then:

(a) if \(0 \leq R \leq \tilde{R}_2\), then the equilibrium consumer market structure is \(0 < v_r < 1\). Price is continuous in \(\pi_r\), and if \(\alpha > 2\) and \(R > 1\), then it decreases in \(\pi_r\) on \((0, \frac{1}{R})\) and increases in \(\pi_r\) on \([\frac{1}{R}, 1)\). Otherwise, it only decreases in \(\pi_r\).

(b) if \(\tilde{R}_2 < R < \tilde{R}_3\), then:

(i) if \(0 \leq \pi_r < \pi'\), then the equilibrium consumer market structure is \(0 < v_{nr} < v_r < 1\). Price is decreasing in \(\pi_r\).\(^{20}\)

(ii) if \(\pi' \leq \pi_r \leq 1\), then the equilibrium consumer market structure is \(0 < v_r < 1\). If \(R > 1\), then price decreases in \(\pi_r\) on \((\pi', \frac{1}{R})\) and increases in \(\pi_r\) on \([\frac{1}{R}, 1)\). Otherwise, it only decreases in \(\pi_r\).

(iii) the vendor discontinuously hikes price at \(\pi_r = \pi'\).\(^{21}\)

We visually present the essence of Proposition 7 in panels (a) and (b) of Figure 7 where we have chosen a ransom level to satisfy part (b) of the proposition statement. Under that parameter set, \(\pi'\) is approximately 0.42. As can be seen when moving from Region (E) to (A), the vendor still discontinuously raises price which is consistent with our finding in part (c) of Proposition 2. Therefore this inherent incentive we established in which the vendor strategically raises price at a higher level of ransom demand is quite robust to the class of ransomware attacks being studied. Moreover, it is this price hike that sets apart vendor strategies in the presence of a ransomware threat compared to strategies employed in its absence, as we will further see in Section 5.

It is also interesting to contrast models \(RW\) and \(RW-OV\). In a context of patchable vulnerabilities, as the risk factor increases, consumers have greater incentives to patch their systems as a means of protection. This leads to a second discontinuous and strategic price shift by the vendor (this time downward) as it can expand the market when more systems are being patched. We illustrated this point earlier in Region (C) of panels (a) and (b) of Figure

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\(^{20}\)The characterization of \(\pi'\) provided in Lemma A.10 of Section A.3 of the Appendix.

\(^{21}\)The characterizations of \(\tilde{R}_2\) and \(\tilde{R}_3\) are provided in Lemma A.10 of Section A.3 of the Appendix.
Figure 7: Impact of risk factor ($\pi_r$) on the equilibrium market outcome, vendor’s price, size of the market segment willing to pay ransom if hit, endogenous risk level, and expected total ransom paid. The parameter values are $\alpha = 0.8$, $\delta = 0.05$, and $R = 0.43$.

3. On the other hand, for ransomware vectors that do not target patchable vulnerabilities as in model RW-OV, this strategic behavior cannot arise and is absent from Proposition 7.

Furthermore, from panels (c) and (e) of Figure 7, it can be observed that the expected ransom paid increases piecewise in $\pi_r$ even as the segment of consumers willing to pay ransom shrinks with the risk factor. In contrast, as mentioned in part (c) of Proposition 3 and illustrated in panel (e) of Figure 3, under patchable vulnerabilities, the expected ransom paid would decrease in the risk factor over region (C) which corresponds to high risk. The following result formalizes this argument:

**Proposition 8.** Under model RW-OV and the same conditions as Proposition 7,

(a) if $0 \leq R \leq \tilde{R}_2$, then:

(i) if $\alpha \leq 2$, or $\alpha > 2$ and $R \leq 1$, then the expected total ransom paid is increasing in $\pi_r$.

Otherwise, the expected total ransom paid increases in $\pi_r$ on $(0, \frac{1}{R})$ and decreases in $\pi_r$ on $[\frac{1}{R}, 1)$.
(b) if $\tilde{R}_2 < R < \tilde{R}_3$, then:

(i) if $0 \leq \pi_r < \pi'$, then the size of the population willing to pay ransom is constant in $\pi_r$ while the expected total ransom paid increases in $\pi_r$;

(ii) if $\pi' \leq \pi_r \leq 1$ and if $R \leq 1$, then the size of the population willing to pay ransom shrinks in $\pi_r$ while the expected total ransom paid increases. On the other hand, if $R > 1$, then the size of the population willing to pay ransom shrinks in $\pi_r$ while the expected total ransom paid increases in $\pi_r$ for $\pi_r < \frac{1}{R}$ and decreases for $\pi_r \geq \frac{1}{R}$.

(iii) the vendor’s price hike at $\pi_r = \pi'$ reduces usage risk to the extent that the expected total ransom paid decreases at the discontinuity as well.

The vendor’s profit and market size are decreasing in $\pi_r$ on each of the specified intervals above, regardless of $R$.

In the above results, the primary differences between the equilibrium outcomes under the ransomware model with patchable vulnerabilities ($RW$) and the ransomware model with unpatchable vulnerabilities ($RW-OV$) are due to the fact that in the former model, Region (C) arises. As a reminder, this particular region corresponds to all segments being present in the market. In such an equilibrium market structure that is commonly observed in practice, increases in either ransom value or risk allow consumers who were previously willing to pay ransom to transition either to patching or to disregarding the ransom request without patching. In the model with a patchable vulnerability, high valuation consumers have an additional lever to mitigate the impact of the ransomware attack, and this capability, which is not an option when the attack enters via an unpatchable threat, can alter the economics of the situation as we have established above.

5 Multiple Classes of Threats on Patchable Vulnerabilities

In previous sections, we focused primarily on ransomware threats. Typically, a successful ransomware attack is preceded by an attacker’s gaining access to the victim’s digital assets remotely (e.g., via a backdoor). In many instances, this is achieved by first exploiting a vulnerability in the victim’s system. Instead of locking access to assets, there can be alternative payloads (e.g., file destruction, diversion of resources, exfiltration of data, etc.) that the malware delivers once it obtains foothold in a system.
In this section, we extend our main model to account for the possibility of other attacks, potentially occurring in parallel with the ransomware threat. We examine attacks whose exploits share a common vulnerability. For example, EternalRocks and WannaCry are both worms that rely on NSA-leaked tools EternalBlue (for lateral propagation) and DoublePulsar (for backdoor implantation) to exploit a Windows Server Message Block vulnerability. However, WannaCry delivers a ransomware payload whereas EternalRocks does not (Ng 2017). In fact, there are over a dozen known large-scale malware campaigns that weaponized EternalBlue to facilitate lateral spread (Keshet 2020). Thus, through proper patching, consumers can protect against all of these threats.

We aim to explore how the presence of ransomware alters software firm strategies and market outcomes even when it is among other threats. For a consumer of type \(v\), we fix the payload impact from the attack to be the same, i.e., \(\alpha v\), for both ransomware and non-ransomware attacks. This enables us to tease out the effect of having a ransom-paying option associated with one of the threats without creating a confounding effect due to differences in payload. This is without a loss in generality in that our model can easily accommodate different payload magnitudes via a transformation.\(^{22}\) Given that the same vulnerability is exploited, we assume interdependent risks characterized by network externalities under all threats. However, we parameterize the risk factor associated with a non-ransomware payload uniquely as \(\pi_n\) such that a successful non-ransomware attack occurs with probability \(\pi_n u(\sigma)\). By permitting \(\pi_r\) and \(\pi_n\) to vary freely, we can examine different attack profiles.

We denote this model of multiple threats with \(MT\) and express the consumer’s utility function as follows:

\[
U_{MT}(v, \sigma) \triangleq \begin{cases} 
  v - p - c_p & \text{if } \sigma(v) = (B, P); \\
  v - p - \pi_r u(\sigma)(R + \delta\alpha v) - \pi_n u(\sigma)\alpha v & \text{if } \sigma(v) = (B, NP, R); \\
  v - p - (\pi_r + \pi_n) u(\sigma)\alpha v & \text{if } \sigma(v) = (B, NP, NR); \\
  0 & \text{if } \sigma(v) = (NB),
\end{cases}
\]

where \(u(\sigma) \triangleq \int_V \mathbb{1}_{\{\sigma(v) \in \{(B, NP, R), (B, NP, NR)\}\}} dv\) is the size of the unpatched adopting population in the presence of the security threat which facilitates both attacks. Similar to before, we assume that \(\delta \in [0, 1], \pi_r, \pi_n \in [0, 1], c_p \in (0, 1), R \in [0, \infty), \) and \(\alpha \in (0, \infty).\)

We first explore, in Section 5.1, two special cases which serve to highlight the difference between having ransomware present versus absent from the threat landscape. By comparing and contrasting these special cases, in the simplest form, our analysis can inform vendors

\(^{22}\)If the payload impact from the non-ransomware attack is \(\alpha_n v\) with \(\alpha_n \neq \alpha\), then we can define \(\bar{\pi}_n = \pi_n \alpha_n / \alpha\), ensuring that \(\pi_n \alpha_n = \bar{\pi}_n \alpha\) and we transform the model into one with equivalent payloads from both types of attacks and risk factor \(\bar{\pi}_n\) for the non-ransomware attack.
and governments on the relative impact of ransomware. Then, in Section 5.2, we examine the robustness of our findings by analyzing practical situations where a mix of ransomware and non-ransomware threats are being faced.

5.1 Special Cases

We begin by exploring the differences in market dynamics between ransomware and non-ransomware threats in the presence of negative security externalities by comparing and contrasting two particular scenarios:

- **Scenario 1**: only a ransomware threat is present. In this special case, \( \pi_n = 0 \) and \( MT \) becomes equivalent to the main model in Section 3, i.e., \( U_{MT} \equiv U_{RW} \).

- **Scenario 2**: only a non-ransomware threat is present. In this special case, \( \pi_r = 0 \) and \( MT \) becomes equivalent to a benchmark model (henceforth denoted \( BM \)) specified in August and Tunca (2006). That is, \( U_{MT} \equiv U_{BM} \).

In this way, model \( MT \) is a generalization that integrates ransomware and non-ransomware threats while preserving the integrity of the existing models that inform the respective components.

In order for the comparisons we make to be meaningful, we hold \( \pi_r + \pi_n = \pi \) as constant. Thus, under scenario 1, we specify \( \pi_r = \pi \) and \( \pi_n = 0 \), whereas under scenario 2, we specify \( \pi_r = 0 \) and \( \pi_n = \pi \). By comparing outcomes between the two scenarios given a common risk factor, we focus in both cases on similar attack vectors for infiltration and spread within networks. As such, any difference in the nature and magnitude of outcomes is attributable to the additional option consumers have in scenario 1 (which is to pay ransom) and the strategic behavior that results from its presence.

Similar to the discussion in Section 3, the more interesting comparative analysis occurs in the case of low \( \delta \) which represents ransomware designed with revenue generation in mind. First, we explore how the vendor’s optimal pricing strategy is different under the two scenarios.

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Their model captures fundamental characteristics of software markets in the presence of malware attacks with no ransom option. Specifically, they study three potential consumer strategies, \( S_{BM} = \{(B,P),(B,NP),(NB,NP)\} \) and capture the network externalities that exist. For a given strategy profile \( \sigma : V \rightarrow S \), the expected utility function for consumer \( v \) is given by:

\[
U_{BM}(v, \sigma) \triangleq \begin{cases} 
    v - p - c_p & \text{if } \sigma(v) = (B,P); \\
    v - p - \pi_n u_{BM}(\sigma)v & \text{if } \sigma(v) = (B,NP); \\
    0 & \text{if } \sigma(v) = (NB),
\end{cases}
\]

where \( u_{BM}(\sigma) \triangleq \int_V 1_{\{\sigma(v) \in \{(B,NP)\}\}} \, dv \) is the size of the unpatched population under the benchmark case.
Proposition 9. There exist $\tilde{\delta}, \hat{\omega} > 0$ and non-overlapping intervals with bounds satisfying $0 < \tilde{\pi}_L < \tilde{\pi}_M < \hat{\pi}_M < \hat{\pi}_H < 1$ such that if $\delta \leq \tilde{\delta}$ and $R_2 < R < \hat{\omega}$:  

(a) if $0 < \pi < \tilde{\pi}_L$, then $p^*_{RW} = p^*_{BM}$;

(b) if $\tilde{\pi}_M < \pi < \hat{\pi}_M$, then $p^*_{RW} > p^*_{BM}$;

(c) if $\tilde{\pi}_H < \pi < 1$, then $p^*_{RW} < p^*_{BM}$.

Proposition 9 is illustrated in Figure 8. For the ransomware scenario, Figure 8 depicts a cross-section of Figure 1 in the $\pi$-direction at $R = 0.43$. In particular, for $\pi$ ranging from 0 to 1, this slice cuts through Regions (E), (A), and (C) of Figure 1; these region labels are also provided in panel (b) of Figure 8. When the risk factor ($\pi$) is sufficiently small, whether ransomware is present in the landscape or not, no customer has any incentive to patch since expected losses are small relative to the cost of patching. In the ransomware scenario, the unpatched consumer indifferent between paying ransom and not paying ransom ($v_r$) derives strictly positive utility in equilibrium. Thus, the emerging market structure is $0 < v_{nr} < v_r < 1$ (as seen in panel (b) of Figure 8 to the left of $\pi \approx 0.31$). Low-valuation consumers who adopt under the given risk circumstances face expected losses that are small enough that paying ransom is not incentive-compatible for them. Hence, the lowest-valuation adopter under the ransomware scenario is unaffected by small perturbations of the ransom demand, although she is affected by the overall unpatched population size. Since the vendor cares about the entire adopter population, the optimal price, profit, and total market size under both ransomware and benchmark scenarios are the same in this region. This can be seen in Figures 8 and 9. In particular, panels (a) and (b) of Figure 9 show the impact of $\pi$ on the vendor’s profit and equilibrium market size, $M(\sigma^*) \triangleq \int_V \mathbb{1}_{\{\sigma^*(v) \in \{(B,NP,NR),(B,NP,R),(B,P)\}\}} dv$.

As $\pi$ increases into moderate and high ranges, we see differences in pricing strategy (and corresponding market outcomes) between the two threat landscapes. While in both scenarios the price remains piecewise decreasing, the two pricing strategies present jumps at points of discontinuity (corresponding to changes in market structure) that highlight significant differences in the vendor’s approach toward mitigating risk using price. In the benchmark case, when the risk factor first reaches a moderate level ($\pi \approx 0.31$), the endogenous overall total security risk $\pi_u$ in equilibrium has also become relatively high (as can be seen from panel (c) of Figure 9). At such a risk level, the vendor has an incentive to significantly drop his price (a discontinuous reduction, illustrated in panel (c) of Figure 8) in a strategic manner to profitably increase the market size. Facing a larger unpatched population, the

\[^{24}R_2\] appears in Proposition 1 and is defined in Lemma A.6.
highest-valuation consumers opt to patch, which insulates them from the added externality introduced by more lower-valuation consumers joining the market but not patching.

However, in the vicinity of \( \pi \approx 0.31 \), this dynamic does not occur in the ransomware case. In particular, if the vendor were to drop the price, some of the highest-valuation consumers would not patch, even when facing this increased risk. Instead, they would still opt to just pay the ransom and bear the valuation-dependent losses \( \delta \alpha v \). As long as these highest-valuation customers remain unpatched, they continue to impose a negative externality on all other unpatched customers in the market. Because of that, the vendor cannot profitably expand the market through a significant drop in price, and the optimal price in the ransomware case stays above the price of the benchmark case.

In stark contrast to the benchmark case, the vendor in the ransomware case actually has an incentive to hike the price altogether to a higher range as risk increases further. This happens for \( \pi \) between 0.42 and 0.57, as can be seen in panel (c) of Figure 8. This outcome and the trade-offs that drive it have been analyzed in Proposition 2 and the related discussion. This is an important point of contrast specific to ransomware; notably, in the benchmark case, the vendor never discontinuously hikes price when changing the market structure from
Figure 9: Sensitivity of vendor’s profit, consumer market size, endogenous total risk level (πu), and social welfare in equilibrium with respect to the risk factor (π) under both benchmark and ransomware scenarios. The parameter values are \( c_p = 0.12, \alpha = 0.8, \delta = 0.05 \) and \( R = 0.43 \).

\( 0 < v_{nr} < 1 \) to \( 0 < v_{nr} < v_p < 1 \) as the risk factor increases. This is because lower-valuation consumers who do not patch would be strongly impacted since there is not an option to pay ransom in order to mitigate losses. As such, a price increase would hurt the unpatched population even more, leading to a significant drop in market size and lower profits. This can be seen in Figure 10 which examines the vendor’s equilibrium price in the benchmark case over the entire space of patching costs \( (c_p) \) and the effective security loss factor \( (\pi n \alpha) \). Figure 10 illustrates that the only price jump that occurs is when the vendor significantly scales back price as the loss factor exceeds a threshold. At that point, the vendor’s price provides the right incentives for a patching population to emerge in equilibrium.

In a similar way, once the risk factor becomes sufficiently high, then the vendor drops price significantly in the ransomware case as well, even below that seen in the benchmark level. This occurs near \( \pi \approx 0.57 \) in panel (c) of Figure 8. This move expands the market significantly at the low end, inviting a large mass of unpatched customers to join the market with a sizable fraction of them opting to not pay the ransom if hit, as seen in panel (b) of
Figure 10: Equilibrium price for the benchmark (model $BM$) as influenced by the magnitude of the patching cost and effective loss factor.

Figure 8. This behavior introduces significant overall risk to the market which is depicted in panel (c) of Figure 9. As a result of this endogenous increase in aggregate risk, high-valuation consumers find it optimal to protect themselves by patching. Consequently, the resulting consumer market equilibrium outcome is $0 < v_{nr} < v_r < v_p < 1$, which most closely resembles today’s software markets.

Interestingly, panel (a) of Figure 9 shows that, once the risk factor is high enough ($\pi > 0.31$), the vendor is strictly better off in scenario 2 (benchmark) than in scenario 1 (ransomware). As $\pi$ increases from 0.31 to 0.42, under model $RW$, the market keeps shrinking even though the price is dropping. In contrast, under model $BM$, the big drop in price leads to a significant expansion in market size. This enables the vendor to obtain higher profits. When $\pi$ ranges from 0.42 to 0.57, under $RW$, the vendor strategically and significantly increases price. As $\pi$ increases through this region, the vendor is better off letting usage shrink rather than lowering the price. Moreover, the unpatched group that pays ransom is highly elastic to risk in this region. On the other hand, under $BM$, the overall
population remains relatively stable (only slightly decreasing), as the presence of security
risk alters the sizes of the patched and unpatched groups in opposite directions in a bal-
anced way. With prices relatively inelastic in this region, the profit under the benchmark is
superior.

When the risk factor is even higher ($\pi > 0.57$), the gap between the two profit levels is
shrinking but the profit under the benchmark scenario still dominates. Because there are
two unpatched groups in the ransomware scenario (those who pay ransom and those who do
not), efforts to increase adoption via lower prices will be associated with higher aggregate risk
($\pi u$). Thus, the reduction in price must be larger for users to bear these risks making it also
much lower than the price employed in the benchmark case. This makes it difficult for the
vendor to be as profitable when facing ransomware. We also note that the vendor’s profit is
less elastic with respect to risk under the benchmark when $\pi \in (0.31, 0.57)$, but the ordering
changes for higher risk levels ($\pi > 0.57$). Comparing the two scenarios, when the threat risk
factor is moderate, the vendor would have a stronger inventive to improve security under
ransomware threats, but the opposite holds when facing a high risk factor. Taken altogether,
consumers in today’s ransomware threat landscape should be more mindful of the risks of
remaining unpatched not only because of the increased externality arising from a sizable
uncharted population but also because of potentially decreased incentives by the vendor to
produce secure software.

Lastly, we study the impact of ransomware on social welfare. Under model $RW$, we denote
the expected aggregate security attack losses incurred by unpatched users who do not pay
ransom by $NL_{RW} \triangleq \int_V 1_{\{\sigma^*(v) = (B,NP,NR)\}} \pi u(\sigma^*) \alpha v dv$. Similarly, we denote the expected ag-
gregate losses incurred by unpatched users who pay ransom by $RL_{RW} \triangleq \int_V 1_{\{\sigma^*(v) = (B,NP,R)\}} \times
\pi u(\sigma^*)(R+\delta \alpha v) dv$, and the aggregate expected patching costs by $PL_{RW} \triangleq \int_V 1_{\{\sigma^*(v) = (B,P)\}} \times
\alpha v dv$. Summing these components, the total expected security-related losses can be ex-
pressed as: $L_{RW} \triangleq NL_{RW} + RL_{RW} + PL_{RW}$. Social welfare is then given by $W_{RW} \triangleq \int_V 1_{\{\sigma^*(v) \in \{(B,NP,NR),(B,NP,R),(B,P)\}\}} \pi u(\sigma^*) \alpha v dv - L_{RW}$. Similarly, for the benchmark case, we de-
fine $NL_{BM} \triangleq \int_V 1_{\{\sigma^*(v) = (B,NP)\}} \pi u(\sigma^*) \alpha v dv$, $PL_{BM} \triangleq \int_V 1_{\{\sigma^*(v) = (B,P)\}} \alpha v dv$, $L_{BM} \triangleq NL_{BM} +
PL_{BM}$, and $W_{BM} \triangleq \int_V 1_{\{\sigma^*(v) \in \{(B,NP),(B,P)\}\}} \pi u(\sigma^*) \alpha v dv - L_{BM}$. The presence of an option to pay
ransom can have a significant effect on the market structure, impacting welfare in complex
ways, which we illustrate in the next result.

**Proposition 10.** There exist $\tilde{\delta}, \tilde{\omega} > 0$ and non-overlapping intervals with bounds satisfying
$0 < \tilde{\pi}_L < \tilde{\pi}_M < \tilde{\pi}_M < \tilde{\pi}_H < 1$ such that if $\delta \leq \tilde{\delta}$, $\alpha > 2c_p + 1$, and $R_2 < R < \tilde{\omega}$:

(a) if $0 < \pi < \tilde{\pi}_L$, then $W_{RW} > W_{BM}$;

(b) if $\tilde{\pi}_M < \pi < \tilde{\pi}_M$, then $W_{RW} < W_{BM}$;
(c) if $\bar{\pi}_H < \pi < 1$, then $W_{RW} > W_{BM}$.

When risk is sufficiently low, consumers do not have strong incentives to patch. Whether they have the option to pay ransom or not, consumers remain unpatched and, as discussed before, the optimal price and market size are identical under the two scenarios. This can be see in panels (b) and (c) of Figure 9 for $\pi < 0.31$. The only difference is that, in the ransomware scenario, unpatched consumers with higher valuations counter the potentially high valuation-dependent losses by paying ransom. Therefore, the overall losses are lower in the ransomware case, which leads to higher social welfare as can be seen in panel (d) of Figure 9.

Social welfare is also higher in the ransomware scenario when risk is high ($\pi > 0.57$) but for a different reason. In this region, high-valuation consumers patch under both scenarios. However, in the face of ransomware, the vendor employs a significantly lower price and achieves a larger market size, including a larger unpatched population. Moreover, some of the unpatched consumers are able to reduce their losses by paying ransom instead. Combining these two effects leads to higher social welfare under ransomware in comparison to the benchmark scenario. Notably, in the high risk region, the vendor’s interests are not aligned with the scenario that yields higher social welfare. Instead, the vendor actually prefers that consumers do not have the recourse of paying ransom. This in turn places additional pressure on consumers, leading to a reduced unpatched population, and ultimately enabling the vendor to charge a higher price.

When the risk is within an intermediate range ($\pi$ between 0.31 and 0.57 in Figure 8), the vendor sets a higher price when ransomware is present compared to the benchmark scenario, resulting in a market size that is significantly lower, as discussed above. While most (and sometimes all) unpatched consumers pay ransom and nobody patches, the significant difference in market size under models $RW$ and $BM$ ensures that social welfare is higher in the latter case, also matching the vendor’s scenario preference.

We next turn our attention to when residual losses ($\delta$) are high, which has been representative of politically-motivated ransomware attacks such as WannaCry (Greenberg 2018). At the end of Section 3.3.2, in Proposition 5, we established that in the ransomware scenario, under high $\delta$, nobody pays ransom in equilibrium. Connecting this outcome to the discussion in this section, the ransomware and benchmark scenarios essentially converge once $\delta$ exceeds a threshold. In other words, the remaining feasible strategies for users match in the scenarios, which gives rise to equivalent equilibrium measures in both cases. In particular, if $\delta > 1 - \frac{\pi}{\tilde{\pi}}$, then $p^*_R = p^*_B$. And consequently, $\Pi^*_R = \Pi^*_B$ and $W_{RW} = W_{BM}$.

Viewed differently, if politically-motivated hackers are utilizing ransomware attacks with high residual losses, then they need not employ ransomware at all; traditional attacks result
in equivalent outcomes. However, given that high $\delta$ ransomware is actually deployed, this equivalence brings forth the question of whether such ransomware could be even more harmful depending on the motivation of the hacker. For example, if a politically-motivated hacker targeted total losses associated with the software, then lowering $\delta$ to induce some ransom payments would actually be more effective. Panel (a) of Figure 11 illustrates a case where lowering $\delta$ to a medium range, i.e., $\delta \in (0.64, 0.76)$, increases expected losses relative to the higher range, i.e., $\delta > 0.76$.

In contrast, at a higher level of ransom demand, a politically-motivated hacker focused on total expected losses might benefit from a minimal $\delta$ which greatly boosts total expected losses as is illustrated in panel (c) of Figure 11. In this case, the behavior of politically-motivated and economically-motivated hackers actually coincides unlike in the case of a lower ransom demand shown in panel (a). On the other hand, as we discussed earlier, it is not easy to translate politically-motivated to an objective goal. For instance, perhaps politically-motivated could instead mean focused on reducing social welfare. In that case, a high level of residual losses where equivalence with the benchmark is achieved would be more effective at reducing welfare, which is illustrated in panel (d) of Figure 11. This case highlights the
Parts (a-d) of Proposition 1 hold

\[ \pi \]

\[ 0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1 \]

\[ \rho \]

\[ 0 \quad 0.05 \quad 0.1 \quad 0.15 \quad 0.2 \quad 0.25 \quad 0.3 \quad 0.35 \]

\[ \pi_n \]

\[ 0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1 \]

\[ \alpha \]

\[ 0 \quad 0.12, \alpha = 0.8, \delta = 0.05. \]

Figure 12: Robustness of Proposition 1 to multiple classes of threats. The parameter values are \( c_p = 0.12, \alpha = 0.8, \) and \( \delta = 0.05. \)

Inherent difficulty with hacker motivations: one gets polarizing predictions depending on how even a single motivation (such as being political) gets operationalized. Layering on that there are many diverse hacker motivations, this issue gets further compounded. In light of these issues, our paper aims to provide insights across a broader set of these motivations by exploring varying levels of \( R \) and \( \delta \) in different regimes. Panel (b) of Figure 11 underscores the complexity that arises by depicting how a lower boundary value of \( \delta \) is the most effective at reducing welfare for a lower level of ransom demand.

### 5.2 Robustness of Results under Multiple Threats Scenarios

Next, we investigate the robustness of several of our key findings on patch-mitigated ransomware threats with respect to a general scenario with multiple classes of threats. Specifically, we examine to what extent the nature of our comparative statics results in \( R \) and \( \pi_r \) (Propositions 1, 2, 3, 9, and 10) continue to hold when we allow for the concomitant presence of more traditional threats in the security landscape (i.e., moving from model \( RW \) to \( MT \)). Due to the additional complexity of model \( MT \), it is necessary to perform the comparisons using numerical analysis. To better illustrate our results, we define \( \pi \triangleq \pi_r + \pi_n \) as the overall risk level and \( \rho \triangleq \pi_r / \pi \in (0, 1] \) as the prevalence of ransomware in the overall threat landscape.
First, we explore the parameter region \((\pi_r, \pi_n)\) under which parts (a)-(d) of Proposition 1 still hold in essence (i.e., we encounter precisely four regions with the same market structures, in the same sequence with respect to \(R\), and the monotonicity of vendor’s price, market size, and profits with respect to \(R\) is the same in each of these regions as in Proposition 1). As illustrated in Figure 12, the results for Proposition 1 hold for a wide region of \((\pi_r, \pi_n)\), with \(\pi_n\) spanning \([0, 0.31]\) and \(\rho\) spanning the entire interval \((0, 1]\). In particular, in the left portion of this region, risk is induced *predominantly* by non-ransomware threats, with a minority contribution from the ransomware threat \((\rho < 0.5)\). This highlights that ransomware, while necessary in the threat landscape, need not be the most prevalent threat for our results to hold. Moreover, if we focus only on arguably the most intriguing part of Proposition 1, namely part (c) which qualifies the existence of a region in \(R\) where all three consumer segments are present and also vendor’s price, market size, and profits all increase in \(R\), the rage of \((\pi_r, \pi_n)\) where such a region exists under the generalized model \(MT\) (with or without the other parts of Proposition 1 necessarily holding) is actually considerably larger than the region described in Figure 12 which is governed by a stricter requirement.

Similarly, we have numerically investigated and confirmed that the results in Propositions 2 and 3 are robust to the presence of traditional threats. For example, under the same parameters as in Figure 3, the essence of Propositions 2 and 3 is satisfied for \(\pi_n \in [0, 0.30]\). For levels of \(\pi_n\) in this range, the behavior exhibited is similar to that in Figure 3 (hence, we omit a matching illustration for brevity).

Last, we explore the robustness of our results in Propositions 9 and 10. For the comparisons to be sensible, we compare and contrast the general scenario (model \(MT\)) outcomes under parameters \((\pi_r, \pi_n)\) to the outcomes under the benchmark model \((BM)\) with a matching aggregate risk factor of \(\pi\). Controlling for overall risk enables us to tease out how the presence of a ransom paying option alters market dynamics. We explore the differences between outcomes under the two scenarios for a continuum of ratios \(\rho \in (0, 1]\) as well as a continuum of overall risk levels, \(\pi \in (0, 1]\). The results of this exploration are depicted in panels (a) and (b) of Figure 13, with the overall risk level, \(\pi\), on the \(x\)-axis, and prevalence of ransomware threat relative to overall risk, \(\rho\), on the \(y\)-axis.\(^{25}\)

The essence of Propositions 9 and 10 can be visualized by the top, horizontal regions (when \(\rho = 1\)) of panels (a) and (b) of Figure 13. As we move away from a ransomware only scenario toward a mixed-threat scenario \(MT\), we incorporate a risk that is a weighted combination of the individual risks in the \(RW\) and \(BM\) models. As can be seen, the dynamics

\(^{25}\)In Figure 13, we illustrate the analysis for the same parameter set that was used to generate Figures 8 and 9. We also conducted a separate sensitivity analysis on each parameter and found that the nature of the regions depicted remains consistent over a broad range of each parameter’s values.
induced by the presence of a ransom option (at all levels of $\rho > 0$) lead to similar strategic pricing decisions and welfare outcomes as the ones described in Section 5.1. Panel (a) highlights the robustness of Proposition 9. When the overall risk in the market is low, then nobody patches and $MT$ and $BM$ scenarios induce the same pricing and market size. As the overall risk in the market increases, we notice at all levels of $\rho > 0$ the presence of a region where $p^*_MT > p^*_BM$ and this region shrinks in width as $\rho$ decreases; this can be expected since, as $\rho$ approaches 0, scenario $MT$ converges to $BM$. The boundary between the leftmost region and middle region is a straight line, corresponding to the $\pi$ value at which $p^*_BM$ drops (which is independent of the change in $\rho$ characterizing the mix of threats under model $MT$).

As the overall risk in the market grows large, we observe $p^*_MT < p^*_BM$ because the presence of a ransom option provides consumers with a means to mitigate risk. This remains true even when ransomware is not the only threat, provided that the security externality permits expansion at the lower end of the market. In panel (b), we observe the same sequence of welfare ordering as we formally established in Proposition 10 for all levels of $\rho$. For low and high overall risk in the market, scenario $MT$ induces a higher social welfare compared to $BM$, whereas the opposite is true for intermediate levels of overall risk. We have extensively discussed the market forces that lead to these outcomes in our analysis of the $RW$ and $BM$ scenarios in Section 5.1. Similar arguments continue to motivate the outcomes in the generalized model.

A common theme that emerges from our robustness analysis is that ransomware need
not be the sole (or even dominant) form of security attack in the market for our results to be applicable. In particular, some of the effects that we establish are specific to the existence of ransomware, and, moreover, these effects only require a small proportion (or likelihood) of ransomware to be in play to already take hold. Since these effects alter the strategic decisions of the software vendor, even the existence of ransomware in a given software market should warrant careful consideration.

6 Conclusion

With the rise of cryptocurrency-based payment systems, malicious hackers are finding it increasingly profitable to conduct ransomware attacks. Modern ransomware variants have exhibited the capability to spread laterally across unprotected systems leading to large scale reach and damage. In this paper, we study the impact of ransomware attacks on software markets. The presence of ransomware in a software product’s threat landscape can qualitatively change the nature of the consumer market structure that obtains in equilibrium. In particular, by giving victims an opportunity to mitigate their losses by paying ransom, ransomware operators segment the unpatched consumer population into two interdependent tiers. Ransomware directly impacts the ransom-paying consumer segment while indirectly impacting all market segments through the negative security externality that all unpatched users generate. This segmentation of consumer behavior drives unexpected findings. For example, both the equilibrium market size and the vendor’s profit under equilibrium pricing can increase in the ransom demand. Moreover, the vendor’s profit can also increase in the residual loss factor (which is negatively related to the trustworthiness of the ransomware operator). Furthermore, we also show that the expected total ransom paid is non-monotone in the risk of success of the attack, increasing when the risk is moderate in spite of a decreasing ransom-paying population.

In order to properly assess the market changes induced by the option to pay ransom, we also compare and contrast market outcomes in the ransomware case to similar outcomes under a benchmark scenario where consumers do not have the option to mitigate the losses by paying ransom. For intermediate levels of risk, the vendor under the ransomware case restricts software adoption by hiking the price to a significantly high level. This lies in stark contrast to outcomes in the benchmark case where any jump in price as security risk increases will be downward. While in low and high risk settings, social welfare is higher under the ransomware case compared to the benchmark case, it turns out that for intermediate risks levels, it is better from a social standpoint for consumers not to have an option to pay ransom. When risk is high, a vendor in the ransomware case has incentives to set a
significantly lower software price compared to the benchmark case; this pricing behavior leads to greater overall risk for consumers as a mass of low-valuation customers enter the market and choose to remain unpatched. While this market expansion is better for social welfare overall, consumers in today’s world end up bearing more risk in a market with ransomware due to increased unpatched usage.

We expand our study in two dimensions by exploring (i) other variants of ransomware whose attacks are not specific to patchable vulnerabilities (e.g., phishing attacks and zero-day attacks), and (ii) other classical attacks on patchable vulnerabilities in addition to ransomware in a generalized model of multiple, concomitant classes of threats. The first expansion clarifies that while the impact of the ransom amount and risk level on equilibrium measures have a similar nature, some important findings are specific to the presence of ransomware risk related to patchable vulnerabilities. For example, the opportunity highlighted where price, market size and profits can all increase in the ransom amount hinges on the impact to patching incentives. Similarly, while strategic price hikes can be observed in the presence of ransomware whether the vulnerability is patchable or not, a strategic price drop that is observed with patchable vulnerabilities requires a higher incentive to patch that occurs in higher risk regions. Overall, the insights we obtain in our primary study are quite robust to generalized attacks in a wide range of scenarios, including threat landscapes where ransomware has only a small presence. Said differently, a little ransomware in the risk profile can be quite influential.

One may desire to model a specific type of hacker motivation and endogenize that hacker’s decisions, in hopes of yielding insights into how the vendor and consumers strategically interact with the hacker. There are two issues that arise. First, our model captures security externalities which cause a significant increase in complexity, e.g., the consumer market threshold characterization is governed by a nonlinear system and the thresholds themselves are the roots of higher order polynomials. Layering the vendor’s pricing optimization problem on top of this foundation already requires asymptotic analysis. Adding more decision variables exacerbates this complexity issue and makes it impossible to characterize the equilibrium across a significant portion of the parameter space. Therefore, such an effort would heavily rely on numerical analysis. Second, one has to give up on other common hacker motivations in order to make that specification which results in a fundamental loss in model generality and applicability. That is, vendors and consumers are facing a diverse mix of hackers which makes it difficult to apply insights drawn from an analysis of a single hacker class. This becomes even more salient if one contrasts the incentives of a profit-motivated hacker to a state actor; their approach to residual losses would essentially be bipolar. The advantage of our model lies in its ability to examine any \( R \) and \( \delta \) that could result from
any practically-relevant model of hacker behavior - such a model would ultimately need to prescribe weights across hacker “types” (motivations) which would then result in a large range of expectations for ransoms and residual losses. This is exactly what our model can examine. Our intention in this research is to inform vendors and consumers on strategies and market outcomes in the presence of a generic ransomware threat, without zooming in on one particular type of hacker. An interesting and rich direction for future research would be a focused exploration of specific markets that are characterized primarily by a single hacker motivation (whether that is profit, disruption, etc.). For such markets, studying how a hacker specifies ransom amounts and residual damages may lead to further insights.

References


Appendix

A.1 Consumer Market Equilibrium

Lemma A.1. The complete threshold characterization of the consumer market equilibrium is as follows:

(I) \(0 < v_{nr} < 1\), where \(v_{nr} = \frac{\pi_r\alpha-1+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}\):

(A) \(p < 1\)
(B) \(R \geq \alpha(1-\delta)\)
(C) \(1 + \pi_r\alpha \leq 2c_p + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}\)

(II) \(0 < v_{nr} < v_p < 1\), where \(v_{nr}\) is the most positive root of the cubic \(f_1(x) \triangleq \pi_r\alpha x^3 + (1 - (c_p + p)\pi_r\alpha)x^2 - 2px + p^2\) and \(v_p = v_{nr} + \frac{v_{nr} - p}{\pi_r\alpha v_{nr}}\):

(A) \(c_p\alpha(R - c_p\alpha(1 - \delta))((1 - \delta)^2 \leq R^2(R - \alpha(c_p + p)(1 - \delta))\pi_r\)
(B) \(R - c_p\alpha(1 - \delta) > 0\)
(C) \((-1 + c_p + p)\pi_r\alpha < -c_p + c_p^2\)

(III) \(0 < v_{nr} < v_r < 1\), where \(v_{nr} = \frac{\pi_r\alpha-1+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}\) and \(v_r = \frac{R}{\alpha(1-\delta)}\):

(A) \(-2R\pi_r + (1 - \delta)(-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}) < 0\)
(B) \(R < \alpha(1-\delta)\)
(C) \(2c_p\alpha + (R + \alpha\delta)(\sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha}) - (1 + \pi_r\alpha)) \geq 0\)

(IV) \(0 < v_{nr} < v_r < v_p < 1\), where \(v_{nr}\) is the most positive root of the cubic \(f_2(x) \triangleq \delta \pi_r\alpha x^3 + (\delta + R\pi_r - (c_p + p\delta)\pi_r\alpha)x^2 - (2\delta + R\pi_r)px + p^2\delta\), \(v_r = \frac{R}{\alpha(1-\delta)}\), and \(v_p = v_{nr} + \frac{v_{nr} - p}{\pi_r\alpha v_{nr}}\):

(A) \(R > p\alpha(1-\delta)\)
(B) \(R^2(R - (c_p + p)\alpha(1 - \delta))\pi_r > -(R - p\alpha(1-\delta))^2\delta(1 - \delta)\)
(C) \(\alpha(c_p - \delta) + \alpha^2\pi_r(c_p\pi_r\alpha + 2p - 2 + \delta(\alpha(p - 1)\pi_r - 3p + 2)) + \sqrt{\pi_r\alpha(\pi_r\alpha + 4p - 2) + 1(\alpha(\pi_r\alpha(c_p + \delta(p - 1)) - c_p + \delta) + \alpha(p - 1)\pi_rR + R)} + R(\pi_r\alpha(\alpha(p - 1)\pi_r - 3p + 2) - 1) < 0\)

(D) Either \((\alpha c_p(\delta - 1)^3(2\pi_r(\alpha p + 1) + \sqrt{(\delta - 1)^2 + 2(\delta - 1)\pi_r(2\alpha(\delta - 1)p + R) + \pi_r^2R^2} \times (\alpha c_p(\delta - 1)^2 + (\delta - 1)R(\pi_r\alpha(c_p + p) + 1) + \pi_r R^2) + (\delta - 1)\pi_r R^2(\pi_r\alpha(c_p + p) + 2) + \)
\[
(\delta - 1)^2 R(2\pi_r \alpha c_p + 3\pi_r \alpha p + 1) + \pi_r^2 R^3 < 0 \] 
and
\[
(\delta + \sqrt{4\alpha(\delta - 1)^2 p\pi_r + (\delta + \pi_r R - 1)^2} < \pi_r R + 1),
\]
or
\[
\left(\frac{\alpha c_p (\delta - 1)^3 (2\pi_r \alpha p + 1) + \sqrt{(\delta - 1)^2 + 2(\delta - 1)\pi_r (2\alpha(\delta - 1)p + R) + \pi_r^2 R^2} \times (-\alpha c_p (\delta - 1)^2 - (\delta - 1) R(\pi_r \alpha (c_p + p) + 1) - \pi_r R^2) + (\delta - 1)\pi_r^2 R^2 (\pi_r \alpha (c_p + p) + 2) + (\delta - 1)^2 R(2\pi_r \alpha c_p + 3\pi_r \alpha p + 1) + \pi_r^2 R^3 > 0 \right) \text{ and } \\
(\pi_r (R - 2\alpha (1 - \delta)p) > -\delta + \sqrt{4\alpha(\delta - 1)^2 p\pi_r + (\delta + \pi_r R - 1)^2} + 1)
\]

\text{(V) } (0 < v_r < 1), \text{ where } v_r = \frac{-1 - R\pi_r + \delta \pi, \alpha + \sqrt{4\delta \pi_r \alpha (p + R\pi_r) + (1 + R\pi_r - \delta \pi, \alpha)^2}}{2\delta \pi, \alpha}.

\text{(A) } p < 1
\]

\text{(B) } c_p \left(\alpha \delta \pi_r + \sqrt{2\pi_r (\alpha \delta (2p - 1) + R)} + \pi_r^2 (\alpha \delta + R)^2 + 1 + \pi_r R + 1 \right) \geq 2(1 - p)\pi_r (\alpha \delta + R)

\text{(C) Either } \left(2\delta + \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} \leq \pi_r \alpha \delta + \pi_r R + 1 \right), \text{ or } \\
\left(2\delta + \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} > \pi_r \alpha \delta + \pi_r R + 1 \right) \text{ and } \\
\frac{\pi_r \alpha \delta^{2} + \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} - R \pi_r - 1}{2\pi_r \alpha \delta} \leq \frac{2\delta p}{\pi_r \alpha \delta^{2} - 2\delta - \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} + \pi_r R + 1}

\text{(VI) } (0 < v_r < v_p < 1), \text{ where } v_r \text{ is the largest positive root of the cubic } f_3 (x) = \alpha^2 \delta^2 \pi_r x^3 + (\alpha \delta (1 + 2R \pi_r - (c_p + p) \delta \pi, \alpha) x^2 + (R^2 \pi_r - 2\alpha \delta (p + (c_p + p) R \pi_r)) x + p^2 \alpha \delta - (c_p + p) R^2 \pi_r \text{ and } v_p = v_r + \frac{v_r - p}{(R + v_r \alpha \delta) \pi_r}.

\text{(A) } \left(\alpha \delta \pi_r + \sqrt{4\alpha \delta \pi_r (p + \pi_r R) + (-\alpha \delta \pi_r + \pi_r R + 1)^2 + \pi_r R - 1} \right) \times \\
\left(\pi_r (\alpha \delta (2c_p + 2p - 1) + R) + \sqrt{4\alpha \delta \pi_r (p + \pi_r R) + (-\alpha \delta \pi_r + \pi_r R + 1)^2} - 1 \right) + \\
2 \left(\alpha \delta \pi_r - 2\alpha \delta p \pi_r + \sqrt{4\alpha \delta \pi_r (p + \pi_r R) + (-\alpha \delta \pi_r + \pi_r R + 1)^2} - R \pi_r - 1 \right)^2 > 0

\text{(B) } \frac{\alpha \delta \pi_r + \sqrt{4\alpha \delta \pi_r (p + \pi_r R) + (-\alpha \delta \pi_r + \pi_r R + 1)^2} - R \pi_r - 1}{2\alpha \delta \pi_r} > p

\text{(C) Either } \left(\pi_r \alpha c_p + \delta^2 \geq \alpha \delta \pi_r (c_p + p) + \delta + \pi_r R \right), \text{ or } \\
\left(\pi_r \alpha c_p + \delta^2 \leq \alpha \delta \pi_r (c_p + p) + \delta + \pi_r R \right) \text{ and } \\
\left(\frac{R}{\alpha (1 - \delta)} \leq p \right) \text{ or } \\
\left(\frac{R}{\alpha (1 - \delta)} > p \text{ and } \pi_r R^2 (\alpha (\delta - 1) (c_p + p) + R) \leq (\delta - 1) \delta (\alpha (\delta - 1) p + R)^2 \text{ and } \frac{R}{\alpha - \alpha \delta} < c_p + p \right) \right)\right)
Proof of Lemma A.1: First, we establish the general threshold-type equilibrium structure. Given the size of unpatched user population \( u \), the net payoff of the consumer with type \( v \) for strategy profile \( \sigma \) is written as

\[
U_{RW}(v, \sigma) \triangleq \begin{cases} 
  v - p - c_p & \text{if } \sigma(v) = (B, P); \\
  v - p - \pi_r u(\sigma)(R + \delta \alpha v) & \text{if } \sigma(v) = (B, NP, R); \\
  v - p - \pi_r \alpha u(\sigma)v & \text{if } \sigma(v) = (B, NP, NR); \\
  0 & \text{if } \sigma(v) = (NB, NP),
\end{cases}
\]  

(A.1)

where

\[
u_{RW}(\sigma) \triangleq \int_v \mathbb{1}_{\{\sigma(v) \in \{(B, NP, R), (B, NP, NR)\}\}} dv.
\]  

(A.2)

Note \( \sigma(v) = (B, P) \) if and only if

\[
v - p - c_p \geq v - p - \pi_r u(\sigma)(R + \delta \alpha v) \iff v \geq \frac{c_p - R \pi_r u(\sigma)}{\delta \pi_r \alpha u(\sigma)}, \text{ and}
\]

\[
v - p - c_p \geq v - p - \pi_r \alpha u(\sigma)v \iff v \geq \frac{c_p}{\pi_r \alpha u(\sigma)}, \text{ and}
\]

\[
v - p - c_p \geq 0 \iff v \geq c_p + p,
\]

which can be summarized as

\[
v \geq \max \left( \frac{c_p - R \pi_r u(\sigma)}{\delta \pi_r \alpha u(\sigma)}, \frac{c_p}{\pi_r \alpha u(\sigma)}, c_p + p \right).
\]  

(A.3)

By (A.3), if a consumer with valuation \( v_0 \) buys and patches the software, then every consumer with valuation \( v > v_0 \) will also do so. Hence, there exists a threshold \( v_p \in (0, 1) \) such that for all \( v \in \mathcal{V} \), \( \sigma^*(v) = (B, P) \) if and only if \( v \geq v_p \). Similarly, \( \sigma(v) \in \{(B, P), (B, NR), (B, R)\} \), i.e., the consumer of valuation \( v \) purchases one of these alternatives, if and only if

\[
v - p - c_p \geq 0 \iff v \geq c_p + p, \text{ or}
\]

\[
v - p - \pi_r \alpha u(\sigma)v \geq 0 \iff v \geq \frac{p}{1 - \pi_r \alpha u(\sigma)}, \text{ or}
\]

\[
v - p - \pi_r u(\sigma)(R + \delta \alpha v) \geq 0 \iff v \geq \frac{p + R \pi_r u(\sigma)}{1 - \delta \pi_r \alpha u(\sigma)},
\]

which can be summarized as

\[
v \geq \min \left( c_p + p, \frac{p}{1 - \pi_r \alpha u(\sigma)}, \frac{p + R \pi_r u(\sigma)}{1 - \delta \pi_r \alpha u(\sigma)} \right).
\]  

(A.4)

Let \( 0 < v_1 \leq 1 \) and \( \sigma^*(v_1) \in \{(B, P), (B, NR), (B, R)\} \), then by (A.4), for all \( v > v_1 \), \( \sigma^*(v) \in \{(B, P), (B, NR), (B, R)\} \), and hence there exists a \( v \in (0, 1] \) such that a consumer
with valuation $v \in \mathcal{V}$ will purchase the software if and only if $v \geq v_p$.

By (A.3) and (A.4), $v \leq v_p$ holds. Moreover, if $v < v_p$, consumers with types in $[v, v_p]$ choose either $(B, NR)$ or $(B, R)$. A purchasing consumer with valuation $v$ will prefer $(B, R)$ over $(B, NR)$ if and only if

$$v - p - \pi_r u(\sigma)(R + \delta \alpha v) \geq v - p - \pi_r \alpha u(\sigma)v \Leftrightarrow v \geq \frac{R}{\alpha(1 - \delta)}. \quad (A.5)$$

Next, we characterize in more detail each outcome that can arise in a consumer market equilibrium, as well as the corresponding parameter regions. For Case (A.1), in which all consumers who purchase choose to be unpatched and not pay ransom, i.e., $0 < v_{nr} < v_p$, based on the threshold-type equilibrium structure, we have $u(\sigma) = 1 - v_{nr}$. We prove the following claim related to the corresponding parameter region in which Case (A.1) arises.

**Claim 1.** The equilibrium that corresponds to case (A.1) arises if and only if the following conditions are satisfied:

$$p < 1 \text{ and } 1 + \pi_r \alpha \leq 2c_p + \sqrt{1 + \pi_r \alpha(-2 + 4p + \pi_r \alpha)}. \quad (A.6)$$

The consumer indifferent between not purchasing at all and purchasing and remaining unpatched, $v_{nr}$, satisfies $v_{nr} - p - \pi_r \alpha u(\sigma)v_{nr} = 0$. To solve for the threshold $v_{nr}$, using $u(\sigma) = 1 - v_{nr}$, we solve

$$v_{nr} = \frac{p}{1 - \pi_r \alpha u(\sigma)} = \frac{p}{1 - \pi_r \alpha(1 - v_{nr})}. \quad (A.7)$$

For this to be an equilibrium, we have that $v_{nr} \geq 0$. This rules out the smaller root of the quadratic as a solution. Given the underlying model assumptions, the other root is strictly positive, so the root characterizing $v_{nr}$ is

$$v_{nr} = \frac{\pi_r \alpha - 1 + \sqrt{1 + \pi_r \alpha(-2 + 4p + \pi_r \alpha)}}{2\pi_r \alpha}. \quad (A.8)$$

For this to be an equilibrium, the necessary and sufficient conditions are that $0 < v_{nr} < 1$, type $v = 1$ weakly prefers $(B, NR)$ to both $(B, R)$ and $(B, P)$.

For $v_{nr} < 1$, it is equivalent to have $p < 1$.

For $v = 1$ to prefer $(B, NR)$ over $(B, P)$, we need $1 \leq \frac{c_p}{\pi_r \alpha(1 - v_{nr})}$. Simplifying, this becomes $1 + \pi_r \alpha \leq 2c_p + \sqrt{1 + \pi_r \alpha(-2 + 4p \pi_r \alpha)}$.

For $v = 1$ to prefer $(B, NR)$ over $(B, R)$, we need $1 \leq \frac{R}{\alpha(1 - \delta)}$. Simplifying, this becomes $R \geq \alpha(1 - \delta)$. The conditions above are given in (A.6). □

Next, for case (II), in which the lower tier of purchasing consumers is unpatched and does not pay ransom while the upper tier patches, i.e., $0 < v_{nr} < v_p < 1$, we have $u = v_p - v_{nr}$. Following the same steps as before, we prove the following claim related to the corresponding
Claim 2. The equilibrium that corresponds to case (II) arises if and only if the following conditions are satisfied:

\[
c_p \alpha (R - c_p \alpha (1 - \delta))(1 - \delta)^2 \leq R^2 (R - \alpha (c_p + p) (1 - \delta)) \pi_r \quad \text{and} \quad R - c_p \alpha (1 - \delta) > 0 \quad \text{and} \quad (-1 + c_p + p) \pi_r \alpha < -c_p + c_p^2. \tag{A.9}
\]

To solve for the thresholds \(v_{nr}\) and \(v_p\), using \(u = v_p - v_{nr}\), note that they solve

\[
v_{nr} = \frac{p}{1 - \pi_r \alpha (v_p - v_{nr})}, \quad \text{and} \quad \tag{A.10}
\]

\[
v_p = \frac{c_p}{\pi_r \alpha (v_p - v_{nr})}. \tag{A.11}
\]

Solving for \(v_p\) in terms of \(v_{nr}\) in (A.10), we have

\[
v_p = v_{nr} + \frac{v_{nr} - p}{\pi_r \alpha v_{nr}}. \tag{A.12}
\]

Substituting this into (A.11), we have that \(v_{nr}\) must be a zero of the cubic equation:

\[
f_1(x) \triangleq \pi_r \alpha x^3 + (1 - \pi_r \alpha (c_p + p)) x^2 - 2px + p^2. \tag{A.13}
\]

To find which root of the cubic \(v_{nr}\) must be, note that the cubic’s highest order term is \(\pi_r \alpha x^3\), so \(\lim_{x \to -\infty} f_1(x) = -\infty\) and \(\lim_{x \to \infty} f_1(x) = \infty\). We find \(f_1(0) = p^2 > 0\), and \(f_1(p) = -c_p \pi_r \alpha p^2 < 0\). Since \(v_{nr} - p > 0\) in equilibrium, we have that \(v_{nr}\) is uniquely defined as the largest root of the cubic, lying past \(p\). Then (A.12) characterizes \(v_p\).

For this to be an equilibrium, the necessary and sufficient conditions are \(0 < v_{nr} < v_p < 1\) and type \(v = v_p\) weakly prefers \((B, P)\) over \((B, R)\). Type \(v = v_p\) preferring \((B, P)\) over \((B, R)\) ensures \(v > v_p\) also prefer \((B, P)\) over \((B, R)\), by (A.3). Moreover, type \(v = v_p\) is indifferent between \((B, P)\) and \((B, NR)\), so this implies that \(v = v_p\) weakly prefers \((B, NR)\) over \((B, R)\). This implies that all \(v < v_{nr}\) strictly prefer \((B, NR)\) over \((B, R)\) by (A.5).

For \(v_p < 1\), first note that from (A.10), we have \(\pi_r \alpha (v_p - v_{nr}) = 1 - \frac{p}{v_{nr}}\) while from (A.11) we have \(\pi_r \alpha (v_p - v_{nr}) = \frac{c_p}{v_p}\). So then solving for \(v_p\), we have

\[
v_p = \frac{c_p v_{nr}}{v_{nr} - p}. \tag{A.14}
\]

Then using (A.14), a necessary and sufficient condition for \(v_p < 1\) to hold is \(v_{nr} > \frac{p}{1 - c_p}\). This is equivalent to \(f_3(\frac{p}{1 - c_p}) < 0\), since \(\frac{p}{1 - c_p} > p\). This simplifies to \((-1 + c_p + p) \pi_r \alpha < -c_p + c_p^2\).

For \(v_p > v_{nr}\), no conditions are necessary since \(v_p\) was defined in (A.12), and \(v_{nr} > p\) by definition of \(v_{nr}\) as the largest root of (A.13).
Similarly, $v_{nr} > 0$, by definition of $v_{nr}$.

To ensure that no consumer has incentive to pay ransom, it suffices to make sure that $v = v_p$ prefers not to pay ransom over paying ransom. By (A.5), we will need $v_p \leq \frac{R}{\alpha(1-\delta)}$. Using (A.14), this is equivalent to $v_{nr}(R - c_p\alpha(1-\delta)) \geq Rp$. If $R - c_p\alpha(1-\delta) \leq 0$, then no $v_{nr}$ would satisfy this condition in equilibrium. Hence, $R - c_p\alpha(1-\delta) > 0$ is a necessary condition and we need $v_{nr} \geq \frac{Rp}{R - c_p\alpha(1-\delta)}$. This simplifies to $f[\frac{Rp}{R - c_p\alpha(1-\delta)}] \geq 0$, which is equivalent to $c_p\alpha(R - c_p\alpha(1-\delta))(1-\delta)^2 \leq R^2(R - \alpha(c_p + p)(1-\delta))\pi_r$. The conditions above are summarized in (A.9). □

Next, for case (III), in which there are no patched users while the lower tier chooses to not pay ransom and the upper tier pays ransom, i.e., $0 < v_{nr} < v_r < 1$, we have $u = 1 - v_{nr}$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (III) arises.

**Claim 3.** The equilibrium that corresponds to case (III) arises if and only if the following conditions are satisfied:

$$p > 0 \text{ and } \alpha(-2R\pi_r + (1-\delta)(-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}) < 0 \text{ and } R < \alpha(1-\delta) \text{ and } 2c_p\alpha + (R + \alpha\delta)\sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha) - (1 + \pi_r\alpha)} \geq 0. \quad (A.15)$$

To solve for the thresholds $v_{nr}$ and $v_r$, using $u = 1 - v_{nr}$, note that they solve

$$v_{nr} = \frac{p}{1 - \pi_r\alpha(1 - v_{nr})}, \quad \text{and} \quad v_r = \frac{R}{\alpha(1 - \delta)}, \quad (A.16)$$

where the expression in (A.17) comes from (A.5).

Solving for $v_{nr}$ in (A.16), we have

$$v_{nr} = \frac{-1 + \pi_r\alpha \pm \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha}. \quad (A.18)$$

Note that $\frac{-1 + \pi_r\alpha - \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha} < p$ while $\frac{-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha} > p$, and since $v_{nr} > p$ in equilibrium, it follows that

$$v_{nr} = \frac{-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha}. \quad (A.19)$$

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_{nr} < v_r < 1$ and that type $v = 1$ weakly prefers $(B, R)$ over $(B, P)$. Type $v = 1$ preferring $(B, R)$ over $(B, P)$ ensures $v < 1$ also prefer $(B, R)$ over $(B, P)$, by (A.3). Moreover, type $v = v_r$ is indifferent between $(B, R)$ and $(B, NR)$, so this implies that $v = v_r$ strictly prefers $(B, NR)$ over $(B, P)$. This implies that all $v < v_{nr}$ strictly prefer $(B, NR)$ over $(B, P)$ as well, again.
The equilibrium that corresponds to case (IV) arises if and only if the following conditions are satisfied:

\[ R > p\alpha(1 - \delta) \text{ and } R^2(R - (c_p + p)\alpha(1 - \delta))\pi_r > -(R - p\alpha(1 - \delta))^2\delta(1 - \delta) \text{ and } \]

\[ \alpha(c_p - \delta) + \alpha^2\pi_r(c_p\pi_r\alpha + 2p - 2) + \delta(\alpha(p - 1)\pi_r - 3p + 2) + \sqrt{\pi_r\alpha(\pi_r\alpha + 4p - 2) + 1(\alpha(\pi_r\alpha(c_p + \delta(p - 1)) - c_p + \delta) + \alpha(p - 1)\pi_r + R + R) + R(\alpha(p - 1)\pi_r - 3p + 2) - 1} < 0 \text{ and } \]

either \[ (\alpha(c_p - \delta - 1)^3(2\pi_r\alpha p + 1) + (\delta - 1)^2 + 2(\delta - 1)\pi_r(2\alpha(\delta - 1)p + R) + \pi_r^2R^2 \times (\alpha(c_p - \delta - 1)^2 + (\delta - 1)R(\pi_r\alpha(c_p + p) + 1) + \pi_r^2R^2) + (\delta - 1)\pi_rR(\pi_r\alpha(c_p + p) + 2) + (\delta - 1)^2R(2\pi_r\alpha c_p + 3\pi_r\alpha p + 1) + \pi_r^2R^3 < 0) \text{ and } \]

\[ (\delta + \sqrt{4\alpha(\delta - 1)^2p\pi_r + (\delta + \pi_rR - 1)^2 < \pi_rR + 1}) \]

or

\[ (\alpha(c_p - \delta - 1)^3(2\pi_r\alpha p + 1) + (\delta - 1)^2 + 2(\delta - 1)\pi_r(2\alpha(\delta - 1)p + R) + \pi_r^2R^2 \times (-\alpha(c_p - \delta - 1)^2 - (\delta - 1)R(\pi_r\alpha(c_p + p) + 1) - \pi_r^2R^2) + (\delta - 1)\pi_rR(\pi_r\alpha(c_p + p) + 2) + (\delta - 1)^2R(2\pi_r\alpha c_p + 3\pi_r\alpha p + 1) + \pi_r^2R^3 > 0) \text{ and } \]

\[ (\pi_r(\pi_r - 2\alpha(1 - \delta)p) > -\delta + \sqrt{4\alpha(\delta - 1)^2p\pi_r + (\delta + \pi_rR - 1)^2 + 1}) \]

To solve for the thresholds \(v_{nr}\) and \(v_p\), using \(u = v_p - v_{nr}\), note that they solve

\[ v_{nr} = \frac{p}{1 - \pi_r\alpha(v_p - v_{nr})}, \text{ and } \]

\[ v_p = \frac{c_p - R\pi_r(v_p - v_{nr})}{\delta\pi_r\alpha(v_p - v_{nr})}, \]

\(\text{Claim 4. The equilibrium that corresponds to case (IV) arises if and only if the following conditions are satisfied:}\)

\[ R > p\alpha(1 - \delta) \text{ and } R^2(R - (c_p + p)\alpha(1 - \delta))\pi_r > -(R - p\alpha(1 - \delta))^2\delta(1 - \delta) \text{ and } \]

\[ \alpha(c_p - \delta) + \alpha^2\pi_r(c_p\pi_r\alpha + 2p - 2) + \delta(\alpha(p - 1)\pi_r - 3p + 2) + \sqrt{\pi_r\alpha(\pi_r\alpha + 4p - 2) + 1(\alpha(\pi_r\alpha(c_p + \delta(p - 1)) - c_p + \delta) + \alpha(p - 1)\pi_r + R + R) + R(\alpha(p - 1)\pi_r - 3p + 2) - 1} < 0 \text{ and } \]

either \[ (\alpha(c_p - \delta - 1)^3(2\pi_r\alpha p + 1) + (\delta - 1)^2 + 2(\delta - 1)\pi_r(2\alpha(\delta - 1)p + R) + \pi_r^2R^2 \times (\alpha(c_p - \delta - 1)^2 + (\delta - 1)R(\pi_r\alpha(c_p + p) + 1) + \pi_r^2R^2) + (\delta - 1)\pi_rR(\pi_r\alpha(c_p + p) + 2) + (\delta - 1)^2R(2\pi_r\alpha c_p + 3\pi_r\alpha p + 1) + \pi_r^2R^3 < 0) \text{ and } \]

\[ (\delta + \sqrt{4\alpha(\delta - 1)^2p\pi_r + (\delta + \pi_rR - 1)^2 < \pi_rR + 1}) \]

or

\[ (\alpha(c_p - \delta - 1)^3(2\pi_r\alpha p + 1) + (\delta - 1)^2 + 2(\delta - 1)\pi_r(2\alpha(\delta - 1)p + R) + \pi_r^2R^2 \times (-\alpha(c_p - \delta - 1)^2 - (\delta - 1)R(\pi_r\alpha(c_p + p) + 1) - \pi_r^2R^2) + (\delta - 1)\pi_rR(\pi_r\alpha(c_p + p) + 2) + (\delta - 1)^2R(2\pi_r\alpha c_p + 3\pi_r\alpha p + 1) + \pi_r^2R^3 > 0) \text{ and } \]

\[ (\pi_r(\pi_r - 2\alpha(1 - \delta)p) > -\delta + \sqrt{4\alpha(\delta - 1)^2p\pi_r + (\delta + \pi_rR - 1)^2 + 1}) \]

To solve for the thresholds \(v_{nr}\) and \(v_p\), using \(u = v_p - v_{nr}\), note that they solve

\[ v_{nr} = \frac{p}{1 - \pi_r\alpha(v_p - v_{nr})}, \text{ and } \]

\[ v_p = \frac{c_p - R\pi_r(v_p - v_{nr})}{\delta\pi_r\alpha(v_p - v_{nr})}, \]
Note that \( v_r \) solves

\[
v_r = \frac{R}{\alpha(1 - \delta)},
\]

where the expression in (A.23) comes from (A.5).

Solving for \( v_p \) in (A.21), we have

\[
v_p = v_{nr} + \frac{v_{nr} - p}{v_{nr} \pi \alpha}.
\]

Substituting this into (A.22), we have that \( v_{nr} \) must be a zero of the cubic equation:

\[
f_2(x) \triangleq \delta \pi \alpha x^3 + (\delta + \pi \alpha (R - \alpha (c_p + \delta p)))x^2 - p(2\delta + R \pi \alpha)x + p^2 \delta.
\]

To find which root of the cubic \( v_{nr} \) must be, note that the cubic’s highest order term is \( \delta \pi \alpha x^3 \), so \( \lim_{x \to -\infty} f_2(x) = -\infty \) and \( \lim_{x \to \infty} f_2(x) = \infty \). We find \( f_2(0) = \delta p^2 > 0 \), and \( f_2(p) = -c_p \pi \alpha p^2 < 0 \). Since \( v_{nr} - p > 0 \) in equilibrium, we have that \( v_{nr} \) is uniquely defined as the largest root of the cubic, lying past \( p \). Then using (A.24), we solve for \( v_p \).

For this to be an equilibrium, the necessary and sufficient conditions are \( 0 < v_{nr} < v_r < v_p < 1 \).

The condition \( v_{nr} > 0 \) is satisfied without further conditions, since \( v_{nr} \) is the largest root of the cubic greater than \( p \) by definition.

For \( v_{nr} < v_r \) to hold, we need \( v_r > p \) and \( f_2(v_r) > 0 \). These conditions are equivalent to \( R > p \alpha (1 - \delta) \) and \( R^2 (R - (c_p + p) \alpha (1 - \delta)) \pi \alpha > -(R - p \alpha (1 - \delta))^2 \delta (1 - \delta) \).

For \( v_r < v_p \) to hold, we need \( \frac{R}{\alpha (1 - \delta)} < v_{nr} + \frac{v_{nr} - p}{\pi \alpha v_{nr}} \) by (A.23) and (A.24). Simplifying, this becomes \( (1 - \delta) \pi \alpha v_{nr}^2 + (\delta - R \pi \alpha) v_{nr} - (1 - \delta) p > 0 \). Then \( v_{nr} \) needs to be larger than the larger root of this quadratic or smaller than the smaller root. The two roots of the quadratic are given by

\[
\frac{-\delta + R \pi \alpha \pm \sqrt{4 \pi \alpha (1 - \delta)^2 (1 - \delta - R \pi \alpha)^2 + 4 (1 - \delta - R \pi \alpha)^2}}{2 \pi \alpha (1 - \delta)}.
\]

If \( v_{nr} \) is larger than the larger root, then a necessary condition is that this larger root is smaller than \( v_r \). On the other hand, if \( v_{nr} \) is smaller than the smaller root, then a necessary condition is that the smaller root is larger than \( p \), since by definition \( v_{nr} > p \).

Consider the first sub-case in which \( v_{nr} \) is larger than the larger root of the quadratic. So then the conditions are

\[
v_{nr} > \frac{-1 + \delta + R \pi \alpha + \sqrt{4 \pi \alpha (1 - \delta)^2 (1 - \delta - R \pi \alpha)^2}}{2 \pi \alpha (1 - \delta)} \quad \text{and} \quad \frac{R}{\alpha (1 - \delta)} \leq p,
\]

either

\[
f_2\left(\frac{-1 + \delta + R \pi \alpha + \sqrt{4 \pi \alpha (1 - \delta)^2 (1 - \delta - R \pi \alpha)^2}}{2 \pi \alpha (1 - \delta)}\right) < 0.
\]

The condition \( R > p \alpha (1 - \delta) \) from \( v_r > p \). Since \( R > p \alpha (1 - \delta) \), a necessary condition is

\[
f_2\left(\frac{-1 + \delta + R \pi \alpha + \sqrt{4 \pi \alpha (1 - \delta)^2 (1 - \delta - R \pi \alpha)^2}}{2 \pi \alpha (1 - \delta)}\right) < 0,
\]

which simplifies to \( \alpha c_p (\delta - 1)^3 (2 \pi \alpha p + 1) + \sqrt{\delta - 1)^2 + 2(\delta - 1) \pi \alpha (2 \alpha (\delta - 1) p + \pi^2 \pi^2 R^2 \times (\alpha c_p (\delta - 1)^2 + (\delta - 1) R (\pi \alpha (c_p + p) + 1) + \pi \alpha R^2) + (\delta - 1) \pi \alpha R^2 (\pi \alpha (c_p + p) + 2) + (\delta - \alpha c_p) < 0.\)
1)^2 R(2\pi r_0 c_p + 3\pi r_0 p + 1) + \pi r_0^2 R^3 < 0. Lastly for this sub-case, we need that the quadratic root 
\[ -1 + \delta + R_{\pi r} + \frac{\sqrt{4\pi r_0 c_p (1 - \delta)^2 + (1 - \delta - R_{\pi r})^2}}{2\pi r_0 (1 - \delta)} < \nu_r = \frac{R}{\alpha(1 - \delta)}. \] This condition simplifies to 
\[ \delta + \sqrt{4\alpha(\delta - 1)^2 p r_{\pi r}} + (\delta + \pi r_0 R - 1)^2 < \pi r_0 R + 1. \] Altogether, these form the first set of conditions in (C) of case (IV).

In the second sub-case in which \( v_{nr} \) is smaller than the smaller root of the quadratic, the necessary and sufficient conditions are that 
\[ \frac{-1 + \delta + R_{\pi r} + \sqrt{4\pi r_0 c_p (1 - \delta)^2 + (1 - \delta - R_{\pi r})^2}}{2\pi r_0 (1 - \delta)} > p \] and 
\[ v_{nr} < \frac{-1 + \delta + R_{\pi r} + \sqrt{4\pi r_0 c_p (1 - \delta)^2 + (1 - \delta - R_{\pi r})^2}}{2\pi r_0 (1 - \delta)}. \] Note that the second condition is equivalent to 
\[ f_2\left(\frac{-1 + \delta + R_{\pi r} + \sqrt{4\pi r_0 c_p (1 - \delta)^2 + (1 - \delta - R_{\pi r})^2}}{2\pi r_0 (1 - \delta)}\right) > 0 \] since \( f_2(x) > 0 \) for any \( x > v_{nr} \). The condition that 
\[ \frac{-1 + \delta + R_{\pi r} + \sqrt{4\pi r_0 c_p (1 - \delta)^2 + (1 - \delta - R_{\pi r})^2}}{2\pi r_0 (1 - \delta)} > p \] simplifies to 
\[ \pi r_0 (R - 2\alpha(1 - \delta)p) > -\delta + \sqrt{4\alpha(\delta - 1)^2 p r_{\pi r}} + (\delta + \pi r_0 R - 1)^2 + 1. \]

The condition that 
\[ f_2\left(\frac{-1 + \delta + R_{\pi r} + \sqrt{4\pi r_0 c_p (1 - \delta)^2 + (1 - \delta - R_{\pi r})^2}}{2\pi r_0 (1 - \delta)}\right) > 0 \] simplifies to \( \alpha c_p (\delta - 1)^3 (2\pi r_0 p + 1) + \sqrt{(\delta - 1)^2 + 2(\delta - 1)\pi r_0 (2\alpha(\delta - 1)p + R) + \pi r_0^2 R^2} \times (-\alpha c_p (\delta - 1)^2 - (\delta - 1)R(\pi r_0 (c_p + p) + 1) - \pi r_0 R^2) + (\delta - 1)\pi r_0 R^2 (\pi r_0 (c_p + p) + 2) + (\delta - 1)^2 R(2\pi r_0 c_p + 3\pi r_0 p + 1) + \pi r_0^2 R^3 > 0. \] Altogether, these form the second set of conditions in (C) of case (IV).

Lastly, we need \( v_p < 1 \). Using (A.24), this simplifies to \( \pi r_0 \nu_{nr}^2 + (1 - \pi r_0) v_{nr} < p. \) Then \( v_{nr} \) needs to be between the two roots of that quadratic, \[ \frac{-1 + \pi r_0 \pm \sqrt{1 - 2\pi r_0 + 4\pi r_0 (\pi r_0)^2}}{2\pi r_0}. \] But note that the smaller of the roots not positive, from \( 0 \leq p \leq 1 \) and \( \pi r_0 > 0 \). Therefore, \( v_{nr} > \frac{-1 + \pi r_0 - \sqrt{1 - 2\pi r_0 + 4\pi r_0 (\pi r_0)^2}}{2\pi r_0} \) is satisfied without further conditions. For \( v_{nr} < \frac{-1 + \pi r_0 + \sqrt{1 - 2\pi r_0 + 4\pi r_0 (\pi r_0)^2}}{2\pi r_0}, f_2\left(\frac{-1 + \pi r_0 + \sqrt{1 - 2\pi r_0 + 4\pi r_0 (\pi r_0)^2}}{2\pi r_0}\right) > 0 \) is a necessary and sufficient condition since 
\[ -1 + \pi r_0 + \sqrt{1 - 2\pi r_0 + 4\pi r_0 (\pi r_0)^2} > p. \] This simplifies to \( \alpha (c_p - \delta) + \alpha^2 \pi r_0 (c_p (\pi r_0 + 2p - 2) + \delta (\alpha (p - 1)\pi r_0 - 3p + 2)) + \sqrt{\pi r_0 (\pi r_0 + 4p - 2) + 1(\alpha (\pi r_0 (c_p + \delta (p - 1)) - c_p + \delta) + \alpha (p - 1)\pi r_0 R + R)(\pi r_0 (\alpha (p - 1)\pi r_0 - 3p + 2) - 1) < 0, \) which is condition (B) of case (IV). Altogether, the conditions above are given in (A.20).

Next, for case (V), in which there are no patched users while all consumers who purchase are unpatched and pay ransom, i.e., \( 0 < v_r < 1 \), we have \( u = 1 - v_r \). Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (V) arises.

**Claim 5.** The equilibrium that corresponds to case (V) arises if and only if the following
conditions are satisfied:

\[ p < 1 \text{ and } \]
\[ c_p \left( \alpha \delta \pi_r + \sqrt{2 \pi_r (\alpha \delta (2p - 1) + R) + \pi_r^2 (\alpha \delta + R)^2 + 1 + \pi_r R + 1} \right) \geq 2(1-p)\pi_r (\alpha \delta + R), \]

and either
\[ 2 \delta + \sqrt{4p \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} \leq \pi_r \alpha \delta + \pi_r R + 1, \text{ or} \]
\[ 2 \delta + \sqrt{4p \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} > \pi_r \alpha \delta + \pi_r R + 1 \text{ and} \]
\[ \frac{\pi_r \alpha \delta + \sqrt{4p \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} - R\pi_r - 1}{2\pi_r \alpha \delta} \leq \frac{2 \delta p}{\pi_r \alpha \delta - 2 \delta - \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2 + \pi_r R + 1}}. \quad (A.26) \]

To solve for the thresholds \( v_r \), using \( u = 1 - v_r \), note it solves
\[ v_r = \frac{p + R\pi_r (1 - v_r)}{1 - \delta \pi_r \alpha (1 - v_r)} \quad (A.27) \]

Then \( v_r \) is one of the two roots of the equation above, \( \frac{-1-R\pi_r + \delta \pi_r \alpha \pm \sqrt{4\delta \pi_r \alpha (p + R\pi_r) + (1 + R\pi_r - \delta \pi_r \alpha)^2}}{2\delta \pi_r \alpha} \).

However, the smaller of the two roots is negative, so \( v_r \) must be the larger of the two roots in equilibrium. Hence, we have
\[ v_r = \frac{-1 - R\pi_r + \delta \pi_r \alpha + \sqrt{4\delta \pi_r \alpha (p + R\pi_r) + (1 + R\pi_r - \delta \pi_r \alpha)^2}}{2\delta \pi_r \alpha}. \quad (A.28) \]

For this to be an equilibrium, the necessary and sufficient conditions are \( p < v_r < 1 \), and no consumer prefers to patch or not pay ransom over paying ransom.

For \( v_r > p \), using (A.28), this simplifies to \( p < 1 \). For \( v_r > p \), using (A.28), this also simplifies to \( p < 1 \). Similarly, \( v_r < 1 \) also simplifies to \( p < 1 \).

For no consumer to strictly prefer paying ransom over paying ransom, it suffices to have type \( v = 1 \) weakly prefer paying ransom to patching. This is given as \( 1 - \pi_r \alpha u(\sigma) \leq 0 \), \( \min(\pi_r \alpha u(\sigma)] \leq 0 \), so that everyone would prefer \( NB, NP \) over \( B, NP, NR \). In this case, no further conditions are needed. On the other hand, if \( 1 - \pi_r \alpha u(\sigma] > 0 \), then we will need the condition \( v_r \leq \frac{p}{1 - \pi_r \alpha (1 - v)} \) for \( v = v_r \) to weakly prefer not buying over buying but not paying ransom.

In the first sub-case, the condition \( v(1 - \pi_r \alpha u(\sigma]) - p < 0 \) simplifies to
\[ 2 \delta + \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} \leq \pi_r \alpha \delta + \pi_r R + 1, \text{ using (A.28)}. \]
In the second sub-case, the conditions $1 - \pi_r \alpha u[\sigma] > 0$ and $v_r \leq \frac{p}{1 - \pi_r \alpha (1 - v_r)}$ simplify to $2\delta + \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} > \pi_r \alpha \delta + \pi_r R + 1$ and

$$\frac{\pi_r \alpha \delta \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} - R\pi_r - 1}{2\pi_r \alpha \delta} \leq -\frac{2\delta p}{\pi_r \alpha \delta - 2\delta - \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} + \pi_r R + 1}.$$  

The conditions above are summarized in (A.26). □

Lastly, for case (VI), in which the top tier patches while lower tier of the market remains unpatched but pays the ransom, i.e., $0 < v_r < v_c < 1$, we have $u = v_p - v_r$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VI) arises.

**Claim 6.** The equilibrium that corresponds to case (VI) arises if and only if the following conditions are satisfied:

$$\left(\alpha \delta \pi_r + \sqrt{4\alpha \delta \pi_r (p + \pi_r R) + (-\alpha \delta \pi_r + \pi_r R + 1)^2} + \pi_r R - 1\right)^2 \times \left(\pi_r (-\alpha \delta (2c_p + 2p - 1) + R) + \sqrt{4\alpha \delta \pi_r (p + \pi_r R) + (-\alpha \delta \pi_r + \pi_r R + 1)^2} - 1\right) + \frac{2\alpha \delta \pi_r}{2\alpha \delta \pi_r} \left(\alpha \delta \pi_r - 2\alpha \delta p \pi_r + \sqrt{4\alpha \delta \pi_r (p + \pi_r R) + (-\alpha \delta \pi_r + \pi_r R + 1)^2} - R\pi_r - 1\right)^2 > 0$$

and

$$\frac{\pi_r \alpha c_p + \delta^2 \geq \alpha \delta \pi_r (c_p + \pi_r R)}{2\alpha \delta \pi_r}, \text{ or } \left(\pi_r \alpha c_p + \delta^2 < \alpha \delta \pi_r (c_p + \pi_r R) \text{ and } \left(\frac{R}{\alpha (1 - \delta)} \leq p \text{ or } \left(\frac{R}{\alpha (1 - \delta)} > p \text{ and } \pi_r R^2 (\alpha (\delta - 1)(c_p + p) + R) \leq (\delta - 1)\delta (\alpha (\delta - 1)p + R)^2 \text{ and } \frac{R}{\alpha - \alpha \delta < c_p + p}\right)\right)\right).$$

(A.29)

To solve for the thresholds $v_r$ and $v_p$, using $u(\sigma) = v_p - v_r$, we solve

$$v_r = \frac{p + R\pi_r u(\sigma)}{1 - \delta \pi_r \alpha u(\sigma)} = \frac{p + R\pi_r (v_p - v_r)}{1 - \delta \pi_r \alpha (v_p - v_r)}, \text{ and }$$

(A.30)

$$v_p = \frac{c_p - R\pi_r u(\sigma)}{\delta \pi_r \alpha u(\sigma)} = \frac{c_p - R\pi_r (v_p - v_r)}{\delta \pi_r \alpha (v_p - v_r)}.$$ (A.31)

Solving for $v_p$ in terms of $v_r$ in (A.30), we have

$$v_p = v_r + \frac{v_r - p}{(R + v_r \alpha \delta)\pi_r},$$

(A.32)

A.11
Substituting this into (A.31), we have that \( v_r \) must be a zero of the cubic equation:

\[
f_3(x) \triangleq \delta^2 \alpha^2 \pi_r x^3 - \alpha \delta (-1 - 2 R \pi_r + (c_p + p) \delta \pi_r \alpha) x^2 + (R^2 \pi_r - 2 \alpha \delta (p + (c_p + p) R \pi_r)) x + p^2 \alpha \delta - (c_p + p) R^2 \pi_r. \quad (A.33)
\]

To find which root of the cubic \( v_r \) must be, note that the cubic’s highest order term is \( \delta^2 \alpha^2 \pi_r x^3 > 0 \), so \( \lim_{x \to -\infty} f_3(x) = -\infty \) and \( \lim_{x \to \infty} f_3(x) = \infty \). Note \( f_3(-\frac{R}{\alpha}) = \alpha \delta (p + \frac{R}{\alpha})^2 > 0 \), \( f_3(p) = -c_p (R + p \alpha \delta)^2 \pi_r < 0 \), and \( f_3(c_p + p) = c_p^2 \alpha \delta > 0 \). Then the root between \( p \) and \( c_p + p \) is the largest positive root of the cubic. Since \( v_r - p > 0 \) in equilibrium, we have that \( v_r \) is uniquely defined as the largest root of the cubic, lying past \( p \). Then using (A.32) to define \( v_p \), we have \( v_p \).

For this to be an equilibrium, the necessary and sufficient conditions are \( 0 < v_r < v_p < 1 \) and no consumer strictly prefers to not pay the ransom over either \((B, P)\) or \((B, R)\).

First, note that \( v_r > p \) implies both \( v_r > 0 \) and \( v_p > v_r \), from (A.32).

For \( v_p < 1 \), using (A.32), this is equivalent to \( \delta \pi_r \alpha v_p^2 + (1 + R \pi_r - \delta \pi_r \alpha) v_r - R \pi_r < p \). For this quadratic in \( v_r \) to be less than a constant, \( v_r \) needs to be between the two roots of the quadratic, \(-1 - R \pi_r + \delta \pi_r \alpha \pm \sqrt{4 \delta^2 \pi_r \alpha (p + R \pi_r) (1 + R \pi_r - \delta \pi_r \alpha)} \). Both roots exist since the radicand is strictly positive.

Note that since \( p \leq 1 \), then \(-1 - R \pi_r + \delta \pi_r \alpha - \sqrt{4 \delta^2 \pi_r \alpha (p + R \pi_r) (1 + R \pi_r - \delta \pi_r \alpha)} \leq p \). Since we already have conditions for \( v_r > p \), this implies that \( v_r \) is larger than the smaller root of the quadratic above.

Then the conditions we need for \( v_p < 1 \) are \( v_r < \frac{-1 - R \pi_r + \delta \pi_r \alpha + \sqrt{4 \delta^2 \pi_r \alpha (p + R \pi_r) (1 + R \pi_r - \delta \pi_r \alpha)}}{2 \delta \pi_r \alpha} \) and \( \frac{-1 - R \pi_r + \delta \pi_r \alpha + \sqrt{4 \delta^2 \pi_r \alpha (p + R \pi_r) (1 + R \pi_r - \delta \pi_r \alpha)}}{2 \delta \pi_r \alpha} > p \). The latter condition is given in (B) of case (VI). With \( \frac{-1 - R \pi_r + \delta \pi_r \alpha + \sqrt{4 \delta^2 \pi_r \alpha (p + R \pi_r) (1 + R \pi_r - \delta \pi_r \alpha)}}{2 \delta \pi_r \alpha} \), it follows that a necessary and sufficient for \( v_r \) is \( \frac{-1 - R \pi_r + \delta \pi_r \alpha + \sqrt{4 \delta^2 \pi_r \alpha (p + R \pi_r) (1 + R \pi_r - \delta \pi_r \alpha)}}{2 \delta \pi_r \alpha} > p \). This is given in (A) of case (VI).

Lastly, we need to ensure that no consumer has an incentive to choose to not patch and not pay ransom. If \( 1 - \pi_r \alpha u(\sigma) \leq 0 \), then \( u(1 - \pi_r \alpha u(\sigma)) \leq 0 \) for all \( v \) so that everyone would weakly prefer \((NB, NP)\) over \((B, NR)\). In this case, we do not need further conditions. Specifically, using (A.32), we have that \( u(\sigma) = v_p - v_r = \frac{v_p - p}{(R + \alpha \pi_r \alpha)} \). So the condition that \( 1 - \pi_r \alpha u(\sigma) \leq 0 \) is equivalent to \( v_r \geq \frac{R + \alpha \pi_r \alpha}{\alpha (1 - \delta)} \). Since \( \frac{R + \alpha \pi_r \alpha}{\alpha (1 - \delta)} > p \), this is equivalent to \( f_3(\frac{R + \alpha \pi_r \alpha}{\alpha (1 - \delta)}) \leq 0 \), which boils down to \( \pi_r \alpha c_p + \delta^2 \geq \alpha \delta \pi_r (c_p + p) + \delta + \pi_r R \).

On the other hand, if \( \pi_r \alpha c_p + \delta^2 < \alpha \delta \pi_r (c_p + p) + \delta + \pi_r R \), then a necessary and sufficient condition for no one to strictly prefer \((B, NR)\) over the other options is for type \( v = v_r \) to weakly prefer \((NB, NP)\) over \((B, NR)\). This would imply that all \( v < v_r \) also have the same preference, from (A.4). Also, since \( v = v_r \) is indifferent between \((NB, NP)\) and \((B, R)\), it follows that \( v = v_r \) weakly prefers \((B, R)\) over \((B, NR)\). Then since only higher-valuation consumers would prefer paying ransom from (A.5), it follows that all
$v > v_r$ would also have the same preference. The condition that $v = v_r$ weakly prefers $(NB, NP)$ over $(B, NR)$ is $v_r \leq \frac{p}{1 - \pi_r \alpha(\frac{p - p_r}{(\pi_r + 2p + \pi_r)\alpha})}$. This simplifies to $v_r \geq \frac{R}{\alpha(1 - \delta)}$.

Now if $\frac{R}{\alpha(1 - \delta)} \leq p$, then no further conditions are needed since $v_r > p$ by definition of $v_r$. On the other hand, if $\frac{R}{\alpha(1 - \delta)} > p$, then a necessary and sufficient condition for $v_r \geq \frac{R}{\alpha(1 - \delta)}$ is for $f_3(\frac{R}{\alpha(1 - \delta)}) \leq 0$ and $\frac{R}{\alpha(1 - \delta)} < c_p$ (since $v_r < c_p$ by construction). This simplifies to $\pi_r R^2 (\alpha(\delta - 1)(c_p + R) \leq (\delta - 1)(\alpha(\delta - 1)p + R)^2$ and $\frac{R}{\alpha - \alpha \delta} < c_p + p$. Altogether, these conditions above are summarized in (A.29). This concludes the proof of the consumer market equilibrium. ■

**Lemma A.2.** The complete threshold characterization of the consumer market equilibrium of the model without patching is as follows:

(I) $(0 < v_{nr} < 1)$, where $v_{nr} = \frac{\pi_r \alpha - 1 + \sqrt{1 + \pi_r \alpha(-2 + 4p + \pi_r \alpha)}}{2\pi_r \alpha}$:

(A) $p < 1$  
(B) $R \geq \alpha(1 - \delta)$

(II) $(0 < v_{nr} < v_r < 1)$, where $v_{nr} = \frac{-1 + \alpha \pi_r + \sqrt{1 + \alpha \pi_r(-2 + 4p + \alpha \pi_r)}}{2\alpha \pi_r}$ and $v_r = \frac{R}{\alpha(1 - \delta)}$:

(A) $R < \alpha(1 - \delta)$  
(B) $(\alpha - \alpha \delta)(1 + \alpha \pi_r + \sqrt{1 + \alpha \pi_r(-2 + 4p + \alpha \pi_r)}) < 2R\alpha \pi_r$

(III) $(0 < v_r < 1)$, where $v_r = \frac{-1 + \pi_r + \sqrt{4 \pi_r \alpha(p + \pi_r R) + (-\pi_r \alpha \pi_r + \pi_r R + 1)^2}}{2\pi_r \alpha}$:

(A) $p < 1$  
(B) Either $2\delta + \sqrt{4 \pi_r \alpha \delta(p + \pi_r R) + (-\pi_r \alpha \pi_r + \pi_r R + 1)^2} \leq \pi_r \alpha \delta + \pi_r R + 1$, or

$$\frac{2\delta + \sqrt{4 \pi_r \alpha \delta(p + \pi_r R) + (-\pi_r \alpha \pi_r + \pi_r R + 1)^2} \leq \pi_r \alpha \delta + \pi_r R + 1 \text{ and } \frac{\pi_r \alpha \delta + \sqrt{4 \pi_r \alpha \delta(p + \pi_r R) + (-\pi_r \alpha \pi_r + \pi_r R + 1)^2} - \pi_r R - 1}{2\pi_r \alpha \delta} \leq -\frac{2p}{\pi_r \alpha \delta - 2\delta - \sqrt{4 \pi_r \alpha \delta(p + \pi_r R) + (-\pi_r \alpha \pi_r + \pi_r R + 1)^2} + \pi_r R + 1}$$

**Proof of Lemma A.2:** From the same argument as the proof of Lemma A.1, we have threshold-type equilibrium structure.

Next, we characterize in more detail each outcome that can arise in a consumer market equilibrium, as well as the corresponding parameter regions. For the case of $0 < v_{nr} < 1$, based on the threshold-type equilibrium structure, we have $u(\sigma) = 1 - v_{nr}$. We prove the following claim related to the corresponding parameter region in which this case arises.
Claim 1. The equilibrium that corresponds to case \( 0 < v_{nr} < 1 \) arises if and only if the following conditions are satisfied:

\[
p < 1 \text{ and } R \geq \alpha(1 - \delta). \tag{A.34}
\]

The consumer indifferent between not purchasing at all and purchasing and remaining unpatched, \( v_{nr} \), satisfies \( v_{nr} - p - \pi_r \alpha u(\sigma) v_{nr} = 0 \). To solve for the threshold \( v_{nr} \), using \( u(\sigma) = 1 - v_{nr} \), we solve

\[
v_{nr} = \frac{p}{1 - \pi_r \alpha u(\sigma)} = \frac{p}{1 - \pi_r \alpha (1 - v_{nr})}. \tag{A.35}
\]

For this to be an equilibrium, we have that \( v_{nr} \geq 0 \). This rules out the smaller root of the quadratic as a solution. Given the underlying model assumptions, the other root is strictly positive, so the root characterizing \( v_{nr} \) is

\[
v_{nr} = \frac{\pi_r \alpha - 1 + \sqrt{1 + \pi_r \alpha (-2 + 4p + \pi_r \alpha)}}{2 \pi_r \alpha}. \tag{A.36}
\]

For this to be an equilibrium, the necessary and sufficient conditions are that \( 0 < v_{nr} < 1 \), type \( v = 1 \) weakly prefers \( (B, NR) \) to both \( (B, R) \).

For \( v_{nr} < 1 \), it is equivalent to have \( p < 1 \).

For \( v = 1 \) to prefer \( (B, NR) \) over \( (B, R) \), we need \( 1 \leq R \frac{\alpha}{\alpha (1 - \delta)} \). Simplifying, this becomes \( R \geq \alpha (1 - \delta) \). The conditions above are given in the lemma. \( \square \)

Next, for the case of \( 0 < v_{nr} < v_r < 1 \), we have \( u = 1 - v_{nr} \). Following the same steps as before, we prove the following claim related to the corresponding parameter region in which this case arises.

Claim 2. The equilibrium that corresponds to case \( 0 < v_{nr} < v_r < 1 \) arises if and only if the following conditions are satisfied:

\[
p > 0 \text{ and } \alpha \left( -2R \pi_r + (1 - \delta)(-1 + \pi_r \alpha + \sqrt{1 + \pi_r \alpha (-2 + 4p + \pi_r \alpha)}) \right) < 0. \tag{A.37}
\]

To solve for the thresholds \( v_{nr} \) and \( v_r \), using \( u = 1 - v_{nr} \), note that they solve

\[
v_{nr} = \frac{p}{1 - \pi_r \alpha (1 - v_{nr})}, \text{ and } \tag{A.38}
\]

\[
v_r = \frac{R}{\alpha (1 - \delta)}, \tag{A.39}
\]

where the expression in (A.39) comes from (A.5).

Solving for \( v_{nr} \) in (A.16), we have

\[
v_{nr} = \frac{-1 + \pi_r \alpha \pm \sqrt{1 + \pi_r \alpha (-2 + 4p + \pi_r \alpha)}}{2 \pi_r \alpha}. \tag{A.40}
\]
Note that \( \frac{-1 + \pi_r \alpha - \sqrt{1 + \pi_r \alpha (2 + 4p + \pi_r \alpha)}}{2\pi_r \alpha} < p \) while \( \frac{-1 + \pi_r \alpha + \sqrt{1 + \pi_r \alpha (2 + 4p + \pi_r \alpha)}}{2\pi_r \alpha} > p \), and since \( v_{nr} > p \) in equilibrium, it follows that

\[
v_{nr} = \frac{-1 + \pi_r \alpha + \sqrt{1 + \pi_r \alpha (2 + 4p + \pi_r \alpha)}}{2\pi_r \alpha}.
\]

(A.41)

For this to be an equilibrium, the necessary and sufficient conditions are \( 0 < v_{nr} < v_r < 1 \). Type \( v = v_r \) is indifferent between \( (B, R) \) and \( (B, NR) \), so this implies that \( v = v_r \) strictly prefers \( (B, NR) \) over \( (B, P) \).

Note \( v_{nr} > 0 \) is satisfied if \( p > 0 \) since \( v_{nr} > p \) under the preliminary model assumptions.

For \( v_{nr} < v_r \), from (A.19) and (A.17), this simplifies to \( \alpha ( -2R \pi_r + (1 - \delta) ( -1 + \pi_r \alpha + \sqrt{1 + \pi_r \alpha ( -2 + 4p + \pi_r \alpha) } ) ) < 0 \).

For \( v_r < 1 \), from (A.17), this simplifies to \( R < \alpha (1 - \delta) \). The conditions above are summarized in the lemma. □

Lastly, for case the case of \( 0 < v_r < 1 \), we have \( u = 1 - v_r \). Following the same steps as before, we prove the following claim related to the corresponding parameter region in which this case arises.

**Claim 3.** The equilibrium that corresponds to case \( 0 < v_r < 1 \) arises if and only if the following conditions are satisfied:

\[
p < 1 \text{ and } 2\delta + \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} \leq \pi_r \alpha \delta + \pi_r R + 1, \text{ or }
\]

\[
2\delta + \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} > \pi_r \alpha \delta + \pi_r R + 1 \text{ and } \frac{\pi_r \alpha \delta + \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} - R\pi_r - 1}{2\pi_r \alpha} \leq \frac{2\delta p}{\pi_r \alpha \delta - 2\delta - \sqrt{4\pi_r \alpha \delta (p + \pi_r R) + (-\pi_r \alpha \delta + \pi_r R + 1)^2} + \pi_r R + 1}. \]

(A.42)

To solve for the thresholds \( v_r \), using \( u = 1 - v_r \), note it solves

\[
v_r = \frac{p + R\pi_r (1 - v_r)}{1 - \delta \pi_r \alpha (1 - v_r)}.
\]

(A.43)

Then \( v_r \) is one of the two roots of the equation above, \( \frac{-1 - R\pi_r + \delta \pi_r \alpha \pm \sqrt{4\delta \pi_r \alpha (p + R\pi_r) + (1 + R\pi_r - \delta \pi_r \alpha)^2}}{2\delta \pi_r \alpha} \). However, the smaller of the two roots is negative, so \( v_r \) must be the larger of the two roots in equilibrium. Hence, we have

\[
v_r = \frac{-1 - R\pi_r + \delta \pi_r \alpha + \sqrt{4\delta \pi_r \alpha (p + R\pi_r) + (1 + R\pi_r - \delta \pi_r \alpha)^2}}{2\delta \pi_r \alpha}.
\]

(A.44)

For this to be an equilibrium, the necessary and sufficient conditions are \( p < v_r < 1 \), and
no consumer prefers to patch or not pay ransom over paying ransom.

For $v_r > p$, using (A.28), this simplifies to $p < 1$. For $v_r > p$, using (A.44), this also simplifies to $p < 1$. Similarly, $v_r < 1$ also simplifies to $p < 1$.

For no consumer to strictly prefer not paying ransom over paying ransom, it suffices to have $v = v_r$ weakly prefer not to buy over buying and not paying ransom (since type $v = v_r$ is indifferent between the option of not purchasing and the option of purchasing, remaining unpatched, and paying ransom). Now if $1 - \pi_r\alpha u[\sigma] \leq 0$, then $v(1 - \pi_r\alpha u[\sigma]) - p < 0$, so that everyone would prefer $(NB, NP)$ over $(B, NP, NR)$. In this case, no further conditions are needed. On the other hand, if $1 - \pi_r\alpha u[\sigma] > 0$, then we will need the condition $v_r \leq p$ for $v = v_r$ to weakly prefer not buying over buying but not paying ransom.

In the first sub-case, the condition $v(1 - \pi_r\alpha u[\sigma]) - p < 0$ simplifies to

$$2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} \leq \pi_r\alpha\delta + \pi_r R + 1,$$

using (A.28).

In the second sub-case, the conditions $1 - \pi_r\alpha u[\sigma] > 0$ and $v_r \leq \frac{p}{1 - \pi_r\alpha(1 - v_r)}$ simplify to

$$2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} > \pi_r\alpha\delta + \pi_r R + 1$$

and

$$\frac{v_r\alpha + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2}}{2\pi_r\alpha} - \frac{2\delta}{\pi_r\alpha - 2\delta - \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2 + \pi_r R + 1}} \leq \frac{2\delta}{\pi_r\alpha - 2\delta - \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2 + \pi_r R + 1}}.$$

The conditions above are summarized in (A.26). □

This completes the proof of the consumer market equilibrium of the model in Section 4 of the paper. ■

Lemma A.3. The complete threshold characterization of the consumer market equilibrium of the benchmark case in Section 5 of the paper is as follows:

(I) $(0 < v_{nr} < 1)$, where $v_{nr} = \frac{\pi_r\alpha - 1 + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha}$:

(A) $p < 1$
(B) $1 + \pi_r\alpha \leq 2c_p + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}$

(II) $(0 < v_{nr} < v_p < 1)$, where $v_{nr}$ is the largest root of the cubic $f(x) = \pi_r\alpha x^3 + (1 - (c_p + p)\pi_r\alpha)x^2 - 2px + p^2$ and $v_p = \frac{c_p v_{nr} - p}{v_{nr} - p}$:

(A) $p < 1$
(B) $c_p + (-1 + c_p + p)\pi_r\alpha < c_p^2$

Proof of Lemma A.3: This follows from the proof of Lemma A.1 with $\delta = 1$. ■
A.2 Proofs of Lemmas Regarding $0 < v_{nr} < v_p < 1$

Lemma A.4. Under the conditions of the focal region \( \left( \text{specifically, } c_p \in (0, 2 - \sqrt{3}) \text{ and } \alpha \in \left( \frac{2}{(1-c_p)^2} - 2, 2(2-c_p)^2 \right) \right) \), if $0 < v_{nr} < v_p < 1$ arises in equilibrium under optimal pricing of the benchmark case, then $v_{nr}^* \geq \frac{1+c_p^2}{2}$ in equilibrium.

Proof of Lemma A.4: Suppose that $0 < v_{nr} < v_p < 1$ is induced. From (A.13), we have that $v_{nr}$ is the largest root of the cubic:

\[
f_1(x) \triangleq \pi_r \alpha x^3 + (1 - \pi_r \alpha (c_p + p))x^2 - 2px + p^2. \tag{A.45}
\]

Then in equilibrium, $p_{II}'$ and $v_{nr}$ must solve $\pi_r \alpha v_{nr}^3 + (1 - \pi_r \alpha (c_p + p))v_{nr}^2 - 2pv_{nr} + p^2 = 0$, where $p_{II}'$ is the equilibrium price of this case. From this, we have that

\[
p_{II}'^* = \frac{1}{2} v_{nr} \left( 2 + \pi_r \alpha v_{nr} \pm \sqrt{\pi_r \alpha (4c_p + \pi_r \alpha v_{nr}^2)} \right). \tag{A.46}
\]

Can $p_{II}' = \frac{1}{2} v_{nr} \left( 2 + \pi_r \alpha v_{nr} + \sqrt{\pi_r \alpha (4c_p + \pi_r \alpha v_{nr}^2)} \right)$? Suppose it were. Then it follows that $p_{II}' > \frac{1}{2} v_{nr} (2 + \pi_r \alpha v_{nr} + \pi_r \alpha v_{nr}) = v_{nr}(1 + \pi_r \alpha v_{nr})$. This is a contradiction, since $v_{nr} > p_{II}'$ in equilibrium (otherwise, some purchasing consumers would derive negative utility upon purchasing). Therefore, we have in equilibrium that

\[
p_{II}' = \frac{1}{2} v_{nr} \left( 2 + \pi_r \alpha v_{nr} - \sqrt{\pi_r \alpha (4c_p + \pi_r \alpha v_{nr}^2)} \right). \tag{A.47}
\]

From (A.47), we have that an expression of the vendor’s equilibrium price as a function of $v_{nr}$ when $0 < v_{nr} < v_p < 1$ is induced in equilibrium.

We will first show that as a function of $v_{nr}$, this price $p(v_{nr})$ increases in $v_{nr}$ to prove monotonicity for any $v_{nr} < 1$ for which $p(v_{nr}) > 0$. Then we will show that $\Pi_{II}(v_{nr})$ increases in $v_{nr}$ for all $v_{nr} < \frac{1+c_p^2}{2}$. By the chain rule, this proves that $v_{nr} \geq \frac{1+c_p^2}{2}$ in equilibrium whenever the interior optimal price of $0 < v_{nr} < v_p < 1$ can be achieved.

Note that $p(v_{nr}) > 0$ for $v_{nr} > \max(0, \frac{-1+c_p \alpha \pi_r}{\alpha \pi_r})$, so consider $v_{nr} \in (\max(0, \frac{-1+c_p \alpha \pi_r}{\alpha \pi_r}), 1)$. Taking the derivative of (A.47) with respect to $v_{nr}$, we find that this derivative is positive for all $v_{nr} \in (\max(0, \frac{-1+c_p \alpha \pi_r}{\alpha \pi_r}), 1)$. Hence, there is a one-to-one relationship between price and $v_{nr}$ on this range.

Next, we show that the vendor’s profit function as a function of $v_{nr}$ is increasing in $v_{nr} \in (\max(0, \frac{-1+c_p \alpha \pi_r}{\alpha \pi_r}), \frac{1+c_p^2}{2})$. Note that $\frac{1+c_p^2}{2} > \frac{-1+c_p \alpha \pi_r}{\alpha \pi_r}$ under the $c_p$ and $\alpha$ assumptions of the focal region. The vendor’s profit function in this case can be written as

\[
\Pi_{II}(v_{nr}) = p_{II}'(1 - v_{nr}), \tag{A.48}
\]
where \( p^*_I \) comes from (A.47). Taking the derivative with respect to \( v_{nr} \) and simplifying, we have that \( \frac{d}{dv_{nr}} \Pi_I(v_{nr}) > 0 \) for all \( v_{nr} \in \max(0, \frac{-1 + c_p \alpha \pi r}{\alpha \pi r}, \frac{1 + c_p^2}{2}) \). This proves that the profit function of this case is increasing for all \( 0 < v_{nr} < \frac{1 + c_p^2}{2} \) for which \( p^*_I(v_{nr}) > 0 \). By the chain rule, since \( p^*_I(v_{nr}) \) is an increasing function of \( v_{nr} \), it also follows that the profit function is increasing in \( p \) for all \( p \) until at least \( v_{nr} = \frac{1 + c_p^2}{2} \). Therefore, if this case is induced in equilibrium under optimal pricing, then \( v_{nr}^* \geq \frac{1 + c_p^2}{2} \) in equilibrium. ■

**Lemma A.5.** Under the conditions of the focal region, if \( 0 < v_{nr} < v_p < 1 \) arises in equilibrium under optimal pricing of the benchmark case, then \( v_{nr} \leq \frac{1 + c_p^2}{2} \).

**Proof of Lemma A.5:** Again, from (A.47), we have that an expression of the vendor’s optimal price as a function of \( v_{nr} \) when \( 0 < v_{nr} < v_p < 1 \) is induced in equilibrium is given by

\[
p^*_I = \frac{1}{2} v_{nr} \left( 2 + \pi_r \alpha v_{nr} - \sqrt{\pi_r \alpha (4c_p + \pi_r \alpha v_{nr}^2)} \right). \tag{A.49}
\]

The vendor’s profit function as a function of \( v_{nr} \) is given by \( \Pi_I(v_{nr}) = p^*_I(v_{nr})(1 - v_{nr}) \).

To prove the lemma, we will show that \( \frac{d}{dv_{nr}} \Pi_I(v_{nr}) < 0 \) for \( v_{nr} \in (\frac{1 + c_p^2}{2}, 1) \).

Using the above expression of \( p^*(v_{nr}) \), we have that

\[
\frac{d}{v_{nr}} \Pi_I(v_{nr}) = \frac{1}{2} \left( (-1 + v_{nr}) \pi_r \alpha v_{nr} \left(-1 + v_{nr} \sqrt{\frac{\pi_r \alpha}{4c_p + \pi_r \alpha v_{nr}^2}} \right) + \right.
\]

\[
( -1 + 2v_{nr}) \left( -2 + \pi_r \alpha v_{nr} + \sqrt{\pi_r \alpha (4c_p + \pi_r \alpha v_{nr}^2)} \right) \right). \tag{A.50}
\]

Now to show that this negative for all \( v_{nr} \in (\frac{1 + c_p^2}{2}, 1) \), we will show that it is negative when \( v_{nr} \) is a convex combination of \( \frac{1 + c_p^2}{2} \) and 1. In particular, substituting \( v_{nr} = w + (1 - w)\frac{1 + c_p^2}{2} \) into (A.50), we will show that this expression is negative for all \( w \in (0, 1) \).

In particular, \( \frac{d}{v_{nr}} \Pi_I(v_{nr}) \bigg|_{v_{nr} = w + (1 - w)\frac{1 + c_p^2}{2}} < 0 \) is equivalent to

\[
\pi_r \alpha (32c_p(w + c_p - wc_p) - (1 + 3w(-1 + c_p) - 3c_p)(1 + w + c_p - wc_p)^2 \pi_r \alpha) < \sqrt{\pi_r \alpha (16c_p + (1 + w + c_p - wc_p)^2 \pi_r \alpha)} \left(8w(1 - c_p) + 8c_p - \pi_r \alpha + \right. \]

\[
(-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p)\pi_r \alpha \right). \tag{A.51}
\]

Now we examine several subcases.

**Subcase 1:** \( w \geq \frac{1}{3} \). First, suppose that \( w \geq \frac{1}{3} \). This implies that \( \pi_r \alpha (32c_p(w + c_p - wc_p) - (1 + 3w(-1 + c_p) - 3c_p)(1 + w + c_p - wc_p)^2 \pi_r \alpha) > 0 \) and \( \left(8w(1 - c_p) + 8c_p - \pi_r \alpha + \right. \]

A.18
\((-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p)\pi_r \alpha) > 0\) as well, for any \(\pi_r, \alpha,\) and \(c_p\) in the focal region. So both the left and right side of the inequality in (A.51) are positive. We isolate the radicand and square both sides. Simplifying and omitting the algebra, this is equivalent to

\[
64c_p(w + c_p - wc_p)^2 > - (4(w(-1 + c_p) - c_p)(w^3(-1 + c_p)^3 + (-3 + c_p)c_p(-1 + 3c_p) + w(-1 + c_p)(1 + c_p)(1 + 11c_p) - w^2(1 - c_p)^2(2 + 15c_p))\pi_r \alpha + w(1 + 3w(-1 + c_p) - 3c_p)(-1 + c_p)(1 + w + c_p - wc_p)^3(\pi_r \alpha)^2) \quad (A.52)
\]

Now viewing the left-hand side as a constant function in \(\alpha\) and the right-hand side as a quadratic function in \(\alpha\), we want to show that the quadratic in \(\alpha\) is smaller than a constant in \(\alpha\). With \(w \geq \frac{1}{3}\), the coefficient on \(\alpha^2\) on the right-hand side is negative. Then it suffices to show that the maximum of that quadratic is less than \(64c_p(w + c_p - wc_p)^2\). Differentiating the right-hand side of the inequality with respect to \(\alpha\) and solving for the maximum, we find that the maximizing \(\alpha\) is negative. Therefore, the right-hand side of (A.52) is maximized at \(\alpha = 0\), which would the right-hand side of the inequality \(0\). Then to show that the inequality above holds for all \(\alpha > 0\), it suffices to show that \(64c_p(w + c_p - wc_p)^2 > 0\), which is true since \(c_p > 0\). \(\square\)

**Subcase 2a:** \(0 \leq w < \frac{1}{3}\) and \(1 - \frac{2}{3(1-w)} \leq c_p < 1\) Now suppose that \(0 \leq w < \frac{1}{3}\) and \(1 - \frac{2}{3(1-w)} \leq c_p < 1\). Going back to the original inequality we want to show, to show that \(\Pi_{v_{nr}} \frac{d}{dv_{nr}} \Pi_{v_{nr}} |_{v_{nr}=w+(1-w)^{1+c_p}} > 0\), we need to show (A.52).

When \(0 \leq w < \frac{1}{3}\) and \(1 - \frac{2}{3(1-w)} \leq c_p \leq 1\), then \(32c_p(w + c_p - wc_p) - (1 + 3w(-1 + c_p) - 3c_p)(1 + w + c_p - wc_p)^2\pi_r \alpha > 0\) and \(8w(1 - c_p) + 8c_p - \pi_r \alpha + (-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p)\pi_r \alpha > 0\). Then similar to Subcase 1, both the left-hand side and right-hand side of the inequality are positive. We isolate the radicand and square both sides. Omitting the algebra, we again want to show the inequality (A.52).

When \(c_p > 1 - \frac{2}{3(1-w)}\), the coefficient on the quadratic \(\alpha\) term of (A.52) is negative, and the same argument as Subcase 1 applies to show that the inequality holds for all \(\alpha > 0\) when \(0 \leq w < \frac{1}{3}\) and \(1 - \frac{2}{3(1-w)} \leq c_p \leq 1\).

On the other hand, if \(c_p = 1 - \frac{2}{3(1-w)}\), then (A.51) reduces to \(\pi_r \alpha(1 - 3w) - (1 - w)\sqrt{\pi_r \alpha(3 + \pi_r \alpha - w(9 + \pi_r \alpha)) / (1-w)} < 0\), which holds for \(\pi_r \alpha > 0\) and \(0 \leq w < \frac{1}{3}\). \(\square\)

**Subcase 2b:** \(0 \leq w < \frac{1}{3}\) and \(0 < c_p < 1 - \frac{2}{3(1-w)}\). Lastly, consider when \(0 \leq w < \frac{1}{3}\) and \(0 < c_p < 1 - \frac{2}{3(1-w)}\).

Firstly, if \(32c_p(w + c_p - wc_p) - (1 + 3w(-1 + c_p) - 3c_p)(1 + w + c_p - wc_p)^2\pi_r \alpha \geq 0\), then \(8w(1 - c_p) + 8c_p - \pi_r \alpha + (-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p)\pi_r \alpha > 0\) holds as well, for any \(\pi_r > 0\), \(\alpha > 0\), \(w \in [0,1]\), and \(c_p \in (0,1)\). In that case, again the inequality (A.51)
would reduce to \((A.52)\), and the same argument from Subcase 1 would apply to show that \((A.51)\) holds.

On the other hand, consider if
\[
32c_p(w + c_p - wc_p) - (1 + 3w(-1 + c_p) - 3c_p)(1 + w + c_p - wc_p)^2 \pi r \alpha < 0.
\]
If
\[
8w(1 - c_p) + 8c_p - \pi r \alpha + (-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p) \pi r \alpha \geq 0,
\]
then \((A.51)\) holds without further conditions since the left-hand side would be negative while the right-hand side would be non-negative.

However, if
\[
8w(1 - c_p) + 8c_p - \pi r \alpha + (-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p) \pi r \alpha < 0,
\]
then we can divide by it and isolate the radicand of \((A.51)\). Squaring both sides and omitting the algebra, this is equivalent to
\[
64c_p(w + c_p - wc_p)^2 < -4(w(-1 + c_p) - c_p)(w^3(-1 + c_p)^3 + (-3 + c_p)c_p(-1 + 3c_p) + w(-1 + c_p)(1 + c_p)(1 + 11c_p) - w^2(1 - c_p)^2(2 + 15c_p)) \pi r \alpha + w(1 + 3w(-1 + c_p) - 3c_p)(-1 + c_p)(1 + w + c_p - wc_p)^3(\pi r \alpha)^2 \tag{A.53}
\]

Since \(0 \leq c_p < 1 - \frac{2}{3(1-w)}\), the coefficient on the quadratic \(\alpha\) term in \((A.53)\) is positive. So to prove the inequality for all \(\alpha\), it suffices to show that the minimum of this quadratic in \(\alpha\) is larger than \(64c_p(w + c_p - wc_p)^2\). Finding the minimizer of the quadratic in \(\alpha\) and comparing it to the lower bound on \(\alpha\) given by
\[
8w(1 - c_p) + 8c_p - \pi r \alpha + (-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p) \pi r \alpha < 0,
\]
we find that the quadratic is minimized at
\[
\alpha = \frac{8(w(1-c_p)+c_p)}{(1+3w(-1+c_p) - 3c_p)(1+w+c_p-wc_p)\pi r}.
\]
The right-hand side of \((A.53)\) evaluated at this \(\alpha\) is indeed larger than \(64c_p(w + c_p - wc_p)^2\) when \(0 < c_p < 1 - \frac{2}{3(1-w)}\) and \(0 \leq w < \frac{1}{3}\). \(\square\)

Then exhausting all sub-cases, it follows that \((A.51)\) holds for all \(w \in [0, 1]\). In particular, this means that
\[
\frac{d}{v_{nr}} \Pi_{II}(v_{nr}) < 0
\]
for any \(v_{nr} > \frac{1+c_p}{2}\). Therefore, \(v_{nr} \leq \frac{1+c_p}{2}\) whenever \(0 < v_{nr} < v_p < 1\) is induced in equilibrium. \(\blacksquare\)
A.3 Characterization of Equilibrium Outcomes

Lemma A.6. There exists a bound \( \tilde{\delta} > 0 \) such that if \( \delta < \tilde{\delta} \) and \( \pi_r > \bar{\pi}_r \) (where \( \bar{\pi}_r \) is defined in the proof of the lemma) then:

(a) if \( 0 \leq R < R_1 \), then the equilibrium consumer market structure is \( 0 < v_r < 1 \);
(b) if \( R_1 \leq R < R_2 \), then the equilibrium consumer market structure is \( 0 < v_r < v_p < 1 \);
(c) if \( R_2 \leq R < R_3 \), then the equilibrium consumer market structure is \( 0 < v_{nr} < v_r < v_p < 1 \);
(d) if \( R \geq R_3 \), then the equilibrium consumer market structure is \( 0 < v_{nr} < v_p < 1 \),

where \( R_1 = \frac{(2-c_p)c_p}{(1-c_p)^2 \pi_r} + \kappa_1(\delta), \ R_2 = \frac{\alpha}{2-c_p} + \kappa_2(\delta), \) and \( R_3 = \frac{\alpha \sqrt{\pi_r} + \sqrt{\alpha(16 c_p + \alpha \pi_r)}}{4 \sqrt{\pi_r}} + \kappa_3(\delta) \).

Proof of Lemma A.6: From Lemma A.1, a unique consumer market equilibrium arises, given a price \( p \). Within each region of the parameter space defined by Lemma A.1, the thresholds \( v_{nr}, v_r, \) and \( v_p \) are smooth functions of the parameters, as well as the vendor’s price \( p \). In the cases where the thresholds are given in closed-form, this is evident. In the cases where these thresholds are implicitly defined as the root of a polynomial, then the smoothness of the thresholds in the parameters follows from the Implicit Function Theorem. Specifically, for each of those cases, the threshold defined was the most positive root \( v_{nr}^* \) (or \( v_r^* \)) of a cubic function of \( v_{nr} \) (or \( v_r \)), \( f(v_{nr}, p) = 0 \). Moreover, the cubic \( f(v_{nr}, p) \) has two local extrema in \( v_{nr} \) and is negative to the left of \( v_{nr}^* \) and positive to the right of it \( (f(v_{nr}^* - \epsilon, p) < 0 \) and \( f(v_{nr}^* + \epsilon, p) > 0 \) for arbitrarily small \( \epsilon > 0 \)). Therefore, \( \frac{\partial f}{\partial v_{nr}}(v_{nr}, p) \neq 0 \) so that the Implicit Function Theorem applies. The thresholds being smooth in \( p \) implies that the profit function for each case of the parameter space defined by Lemma A.1 is smooth in \( p \). In our proofs, we use asymptotic analysis to characterize the equilibrium prices and profits when needed, using Taylor series representations in \( \delta \) of the thresholds, price, and profit expressions. When writing the Taylor series, we will abuse notation by re-using the same notation for the remainder terms throughout the paper. This is just to simplify the notation. The remainder terms for Taylor series of different expressions are not the same.

In the proofs of the lemmas characterizing equilibrium outcomes, we find the profit-maximizing interior solution within the compact closure of each subcase to characterize the conditions under which each of the consumer market outcomes of Lemma A.1 could arise in under optimal pricing and verify that the second-order condition holds for each of these prices under the conditions of their respective regions. We then find regions of the parameter space where there is overlap between the different cases under optimal pricing to find the boundaries between different regions defining equilibrium outcomes. In the cases where a regime switch happens not at a discontinuous price change but from the vendor choosing a boundary price, the boundary price can be found from finding common region boundaries in Lemma A.1.
First, we specify the interior optimal price and vendor’s profit at that interior optimal price for all market outcomes except for $0 < v_{nr} < v_p < 1$ (which is handled separately in Section A.2 of the Appendix).

Given a price $p$, the region of the parameter space defining $0 < v_{nr} < 1$ is given in part (I) of Lemma A.1. For this case, we have

$$v_{nr} = \frac{-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(1 - 2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha}.$$  \hspace{1cm} (A.54)

The profit function in this case is $\Pi_I(p) = p(1 - v_{nr}(p))$. Let $C_I$ be the compact closure of the region of the parameter space defining $0 < v_{nr} < 1$, given in part (I) of Lemma A.1. By the Weierstrass extreme value theorem, there exists $p$ in $C_I$ that maximizes $\Pi_I(p)$. If this $p$ is interior to $C_I$, the unconstrained maximizer satisfies the first-order condition. The Weierstrass extreme value applies for all regions, and we will not state this for other regions.

Differentiating the profit function with respect to $p$ and solving for the positive root of the quadratic, we have that

$$p^*_I = \frac{1}{9} \left(4 - \frac{1}{\pi_r\alpha} - \pi_r\alpha + \frac{\sqrt{1 + \pi_r\alpha(1 - 2 + 4p + \pi_r\alpha)}}{\pi_r\alpha}\right).$$  \hspace{1cm} (A.55)

Given a price $p$, the region of the parameter space defining $0 < v_{nr} < v_r < 1$ is given in part (III) of Lemma A.1. For this case, we have

$$v_{nr} = \frac{-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(1 - 2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha}.$$  \hspace{1cm} (A.56)

The profit function in this case is $\Pi_{III}(p) = p(1 - v_{nr}(p))$. Let $C_{III}$ be the compact closure of the region of the parameter space defining $0 < v_{nr} < v_r < 1$, given in part (III) of Lemma A.1. As in the previous case, there exists $p$ in $C_{III}$ that maximizes $\Pi_{III}(p)$. If this $p$ is interior to $C_{III}$, the unconstrained maximizer satisfies the first-order condition.

Differentiating the profit function with respect to $p$ and solving for the positive root of the quadratic, we have that

$$p^*_{III} = \frac{1}{9} \left(4 - \frac{1}{\pi_r\alpha} - \pi_r\alpha + \frac{\sqrt{1 + \pi_r\alpha(1 - 2 + 4p + \pi_r\alpha)}}{\pi_r\alpha}\right).$$  \hspace{1cm} (A.57)

The profit corresponding to this price for this case is given by:

$$\Pi^*_{III} = \frac{3 + 3\alpha\pi_r - \sqrt{5 + 5\pi_r\alpha(1 - 2 + 4\pi_r\alpha) + 4\sqrt{1 + \pi_r\alpha(1 - 2 + 4p + \pi_r\alpha)}}}{54(\pi_r\alpha)^2} \times \left(-1 + \pi_r\alpha(4 - \pi_r\alpha) + \sqrt{1 + \pi_r\alpha(1 - 2 + 4\pi_r\alpha) + 4\sqrt{1 + \pi_r\alpha(1 - 2 + 4p + \pi_r\alpha)}}\right).$$  \hspace{1cm} (A.58)
Given a price \( p \), the region of the parameter space defining \( 0 < v_{nr} < v_r < v_p < 1 \) is given in part (IV) of Lemma A.1.

From (A.25), we have that \( v_{nr} \) is the largest root of

\[
f_2(x) \triangleq \delta \pi_r \alpha x^3 + (\delta + \pi_r (R - \alpha (c_p + \delta p)))x^2 - p(2\delta + R\pi_r)x + p^2\delta = 0. \tag{A.59}
\]

A generalization of the Implicit Function Theorem gives that \( v_{nr} \) is not only a smooth function of the parameters, but it is also an analytic function of the parameters so that it can be represented locally as a Taylor series of its parameters. More specifically, since \( f_2'(x) \neq 0 \) at the root for which \( v_{nr} \) is defined, there exists a \( \delta_1 > 0 \) such that for \( \delta < \delta_1 \), \( v_{nr} = \sum_{k=0}^{\infty} a_k \delta^k \) for some sequence of coefficients \( \alpha_k \). Substituting \( v_{nr} = \sum_{k=0}^{\infty} a_k \delta^k \) into (A.59), we have that

\[
-a_0(a_0 \pi_r \alpha c_p - a_0 R \pi_r + p R \pi_r) + \sum_{k=1}^{\infty} a_k \delta^k = 0.
\]

Then \( a_0 = 0 \) or \( a_0 = \frac{p R}{R - \alpha c_p} \) are the only solutions for \( a_0 \) that make the first term zero. Now, \( a_0 \neq 0 \), since otherwise \( v_{nr} < p \) for sufficiently low \( \delta \), which cannot happen. So \( a_0 = \frac{p R}{R - \alpha c_p} \). Then substituting \( v_{nr} = \frac{p R}{R - \alpha c_p} \) + \( \sum_{k=1}^{\infty} a_k \delta^k \) into (A.59) and similarly solving for \( a_1 \), we have \( a_1 = \frac{c_p p \alpha^2 (c_p \pi_r \alpha - R \pi_r (c_p + p R \pi_r))}{R (R - c_p \alpha)^3 \pi_r^2} \). Continuing on this way, we have that

\[
v_{nr} = \frac{p R}{R - c_p \alpha} + \frac{c_p p \alpha^2 (-c_p R + c_p^2 \alpha - p R^2 \pi_r) \delta}{R (R - c_p \alpha)^3 \pi_r} - \frac{c_p p \alpha^3 (-c_p R + c_p^2 \alpha - p R^2 \pi_r) (c_p (-2 R + c_p \alpha) (-R + c_p \alpha) + p R^2 (R + c_p \alpha) \pi_r) \delta^2}{R^3 (R - c_p \alpha)^5 \pi_r^2} + \sum_{k=3}^{\infty} a_k \delta^k. \tag{A.60}
\]

The profit function in this case is \( \Pi_{IV}(p) = p(1 - v_{nr}(p)) \). Let \( C_{IV} \) be the compact closure of the region of the parameter space defining \( 0 < v_{nr} < v_r < v_p < 1 \), given in part (IV) of Lemma A.1. There exists \( p \) in \( C_{IV} \) that maximizes \( \Pi_{IV}(p) \). If this \( p \) is interior to \( C_{IV} \), the unconstrained maximizer satisfies the first-order condition.

Substituting (A.60) into the first-order condition and expanding the optimal price as a Taylor series in \( \delta \), we can then characterize the asymptotic expansion of the optimal price in the same way we had done above with \( v_{nr} \). Omitting the algebra, we have that the interior
solution is given by
\[ p^*_V = \frac{R - c_p\alpha}{2R} + \frac{c_p\alpha^2(4c_p + 3R\pi_r)\delta}{8R^3\pi_r} + \frac{c_p\alpha^3(16c_p^3\alpha - 4R^3\pi_r^2 + c_pR^2\pi_r(-18 + 5\pi_r\alpha))}{16R^5(R - c_p\alpha)\pi_r^2} + \sum_{k=3}^\infty a_k\delta^k. \quad (A.61) \]

The profit associated with this price is given by
\[ \Pi^*_V = \frac{R - c_p\alpha}{4R} + \frac{c_p\alpha^2(2c_p + R\pi_r)\delta}{8R^3\pi_r} + \frac{c_p\alpha^3(32c_p^3\alpha - 4R^3\pi_r^2 + 8c_p^2R(-4 + 3\pi_r\alpha) + c_pR^2\pi_r(-24 + 5\pi_r\alpha))}{64R^5(R - c_p\alpha)\pi_r^2} + \sum_{k=3}^\infty a_k\delta^k. \quad (A.62) \]

Given a price \( p \), the region of the parameter space defining \( 0 < v_r < 1 \) is given in part (V) of Lemma A.1.

From (A.28), we have that
\[ v_r = \frac{-1 - R\pi_r + \delta\pi_r\alpha + \sqrt{4\delta\pi_r\alpha(p + R\pi_r) + (1 + R\pi_r - \delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}. \quad (A.63) \]

The vendor’s profit function is given as \( \Pi_V(p) = p(1 - v_r(p)) \). Solving for the first-order condition (and looking at only the positive root) gives
\[ p^*_V = \left( -1 - 2R\pi_r + 4\delta\pi_r\alpha - R^2\pi_r^2 - 2R\delta\alpha\pi_r^2 - (\delta\pi_r\alpha)^2 + (1 + R\pi_r + \delta\pi_r\alpha)\sqrt{1 + \pi_r(2R - \delta\alpha + (R + \delta\alpha)^2\pi_r)} \right) \left( 9\delta\pi_r\alpha \right)^{-1}. \quad (A.64) \]

Substituting (A.64) into the profit function of this case yields the associated maximal profit of this case. Characterizing this profit expression in terms of a Taylor Series expansion,
\[ \Pi^*_V = \frac{1}{4(1 + R\pi_r)} + \sum_{k=1}^\infty a_k\delta^k. \quad (A.65) \]

Given a price \( p \), the region of the parameter space defining \( 0 < v_r < v_p < 1 \) is given in part (VI) of Lemma A.1. From (A.33), we have that \( v_r \) is the largest root of
\[ \delta^2\alpha^2\pi_r x^3 - \alpha\delta(-1 - 2R\pi_r + (c_p + p)\delta\pi_r\alpha)x^2 + (R^2\pi_r - 2\alpha\delta(p + (c_p + p)R\pi_r))x + p^2\alpha\delta - (c_p + p)R^2\pi_r = 0. \quad (A.66) \]

Characterizing the asymptotic expansion of \( v_r \) as we had done for earlier cases, we have
that
\[ v_r = c_p + p - \frac{c_p^2 \alpha \delta}{R^2 \pi_r} + 2 \left( \frac{c_p^3 \alpha^2 + c_p^2 R \alpha^2 \pi_r + c_p^2 p R \alpha^2 \pi_r}{R^4 \pi^2_r} \right) \delta + \sum_{k=3}^{\infty} a_k \delta^k. \]  
\[ (A.67) \]

The profit function for this case is given by \( \Pi_{VI} = p(1 - v_r(p)) \). Assuming an interior solution, the first-order condition gives
\[ p_{VI}^* = \frac{1 - c_p}{2} + \frac{c_p^2 \alpha \delta}{2 R^2 \pi_r} + \sum_{k=2}^{\infty} a_k \delta^k. \]  
\[ (A.68) \]

The corresponding profit is given by
\[ \Pi_{VI}^* = \frac{1}{4} (1 - c_p)^2 + \frac{(1 - c_p) c_p^2 \alpha \delta}{2 R^2 \pi_r} + \sum_{k=2}^{\infty} a_k \delta^k. \]  
\[ (A.69) \]

Now that we have found the interior optimal prices for these regions, we use Lemma A.1 to find conditions under which the interior optimal price for a case lies within the set of conditions defining that case. Omitting the algebra, for sufficiently high \( \pi_r \) (defined momentarily), there is an overlap between the regions in which \( 0 < v_r < 1 \) and \( 0 < v_r < v_p < 1 \) are optimal. There is also an overlap between the regions in which \( 0 < v_r < v_p < 1 \) and \( 0 < v_nr < v_r < v_p < 1 \) are optimal. Hence, in these cases, the threshold in \( R \) that defines their region boundary can be found by equating their profits at their interior optimal price.

Equating (A.65) and (A.69) and solving for \( R \), we have that the boundary in \( R \) can be expressed as
\[ R_1 = \frac{(2 - c_p) c_p}{(1 - c_p) \pi_r} + \sum_{k=1}^{\infty} a_k \delta^k. \]  
\[ (A.70) \]

For \( R < R_1 \), the profit at the interior optimal solution of \( 0 < v_r < 1 \) dominates that of \( 0 < v_r < v_p < 1 \). However, for \( R > R_1 \), we have the reverse being true.

Similarly, equating (A.69) and (A.62) and solving for \( R \), we have that the boundary in \( R \) can be expressed as
\[ R_2 = \frac{\alpha}{2 - c_p} - \left( \frac{\alpha}{2} + \frac{c_p^2}{\pi_r} \right) \delta + \sum_{k=2}^{\infty} a_k \delta^k. \]  
\[ (A.71) \]

For \( R < R_2 \), the profit at the interior optimal solution of \( 0 < v_r < v_p < 1 \) dominates that of \( 0 < v_nr < v_r < v_p < 1 \). However, for \( R > R_2 \), we have the reverse being true. In order for \( R_1 < R_2 \) for sufficiently small \( \delta \), we need to have \( \pi_r > \bar{\pi}_r \) where
\[ \bar{\pi}_r = \frac{(2 - c_p)^2 c_p}{(1 - c_p)^2 \alpha} + \sum_{k=1}^{\infty} a_k \delta^k. \]  
\[ (A.72) \]

To make sure that \( \bar{\pi}_r < 1 \) for sufficiently high \( \delta \), we need \( \alpha > \frac{2}{(1-c_p)^2} - 2 \).
Lastly, note that $0 < v_{nr} < v_r < v_p < 1$ and $0 < v_{nr} < v_p < 1$ share a price boundary. In particular, from Lemma A.1, $c_p\alpha (R - c_p\alpha) \leq R^2 (R - (c_p + p)\alpha)\pi_r$ is the boundary that is shared between them. The boundary price can be found by solving for $p$ in that inequality,

$$p_{\text{boundary}} = \frac{(R - c_p\alpha)(-c_p\alpha + R^2\pi_r)}{R^2\alpha\pi_r}.$$  \hspace{1cm} (A.73)

Under the conditions $R > R_2$, $\pi_r > \bar{\pi}_r$, and $\alpha > \frac{2}{(1-c_p)^2} - 2$, we have that $p_{\text{boundary}} > 0$.

The $R$ boundary between $0 < v_{nr} < v_r < v_p < 1$ and $0 < v_{nr} < v_p < 1$ can be found by equating (A.61) to this boundary price. This $R$ boundary is given by

$$R_3 = \frac{\alpha}{4} + \frac{\sqrt{\alpha(16c_p + \alpha\pi_r)}}{4\sqrt{\pi_r}} + \sum_{k=1}^{\infty} a_k\delta^k. \hspace{1cm} (A.74)$$

For $R < R_3$, the price at the interior optimal solution of $0 < v_{nr} < v_r < v_p < 1$ is less than $p_{\text{boundary}}$. However, for $R > R_3$, we have the reverse being true. From taking the derivative with respect to $R$ in (A.61), we can see that this price is increasing in $R$. Hence, the interior optimal price of $0 < v_{nr} < v_r < v_p < 1$ will hit the boundary price at $R = R_3$.

Consequently, for $R > R_3$, $0 < v_{nr} < v_p < 1$ will dominate $0 < v_{nr} < v_r < v_p < 1$.

At the interior optimal solution of $0 < v_{nr} < v_r < v_p < 1$, $R = R_3$, we have that $v_{nr} = \frac{1}{2} + O(\delta)$. By Lemma A.4, we have that the profit function of $0 < v_{nr} < v_r < v_p < 1$ is increasing in $p$ for all $p$ up until $p_{\text{boundary}}$. Hence $0 < v_{nr} < v_p < 1$ is dominated by $0 < v_{nr} < v_r < v_p < 1$ for $R < R_3$.

For high enough $R$ (for example $R = \alpha$), $0 < v_{nr} < v_r < v_p < 1$ cannot be induced in equilibrium for any price $p$, and any $R$ beyond that point will no longer have $p = p_{\text{boundary}}$. That $R$ value is an upper bound for another bound $\omega$, which defines the point at which $0 < v_{nr} < v_p < 1$ would no longer have its interior optimal price $p_{II}^*$ be constrained by a region boundary with $0 < v_{nr} < v_r < v_p < 1$. Hence, for high enough $R$ (defined as $\omega$),

$$\omega = \arg \max_R \{ R : p_{\text{boundary}}(R) = p_{II}^* \}. \hspace{1cm} (A.75)$$

That $p_{\text{boundary}}$ is increasing in $R$ implies that $R_3 \leq \omega$. For $R_3 > R_2$ for sufficiently low $\delta$, we need $\pi_r\alpha < 2(2 - c_p)^2$. This is implied by $\alpha < 2(2 - c_p)^2$.

Lastly, we will show that under the conditions of this lemma, namely $\pi_r > \bar{\pi}_r$, the interior optimal price of $0 < v_{nr} < v_r < 1$ lies outside of the region defining it (in Lemma A.1). The interior optimal price of $0 < v_{nr} < v_r < 1$ is given in (A.57). Substituting in this price into the conditions defining this case, it follows that a necessary condition for the interior optimal solution to indeed lie in the region defining the case is

$$\pi_r < \frac{-1 + \sqrt{1 + c_p(-2 + 9\alpha) + c_p(-4 + 9c_p - 3\sqrt{1 + c_p(-2 + 9\alpha)})}}{2(-1 + 2c_p)\alpha}. \hspace{1cm} \text{However, from the conditions in the focal region on } c_p \text{ and } \alpha, \text{ we have that } \bar{\pi}_r > \frac{-1 + \sqrt{1 + c_p(-2 + 9\alpha) + c_p(-4 + 9c_p - 3\sqrt{1 + c_p(-2 + 9\alpha)})}}{2(-1 + 2c_p)\alpha}. \hspace{1cm} \text{Hence, for } \pi_r > \bar{\pi}_r, \text{ we cannot have } 0 < v_{nr} < v_r < 1 \text{ induced in equilibrium. The same analysis can}$$

A.26
be done for $0 < v_{nr} < 1$ to rule that case out from being induced in equilibrium for $π_r > π_r$.

This concludes the proof of the lemma. ■

**Lemma A.7.** There exist bounds $δ > 0$ and $ω = c_p - \frac{α}{2}$ such that if $δ < δ$ and $R \in (R_2, ω)$, then:

(a) if $0 ≤ π_r ≤ π$, then the equilibrium outcome is $0 < v_{nr} < v_r < 1$

(b) if $π ≤ π_r ≤ π_2$, then the equilibrium outcome is $0 < v_r < 1$

(c) if $π_2 ≤ π_r ≤ 1$, then the equilibrium outcome is $0 < v_{nr} < v_r < v_p < 1$

where the bounds on $π_r$ are characterized in the proof below.

**Proof of Lemma A.7:** From Lemma A.6, if $R > R_2$ (where $R_2$ is defined in (A.71)), then we can rule out $0 < v_r < v_p < 1$ from arising in equilibrium. Furthermore, from Lemma A.6, $0 < v_{nr} < v_p < 1$ is ruled out of equilibrium for low enough $R$ (for example, $R < R_3$ (where $R_3$ is defined in (A.74))). Similarly, we can also show that $0 < v_{nr} < 1$ is dominated by $0 < v_{nr} < v_r < 1$ for $R < α(1 - δ)$. Hence, we can find an upper bound on the range of $R$ to ensure $0 < v_{nr} < v_p < 1$ and $0 < v_{nr} < 1$ do not arise in equilibrium. Such a bound greater than $\frac{α}{2 - c_p}$ can be found since $R_2 < R_3$ from Lemma A.6 and $\frac{α}{2 - c_p} < α(1 - δ)$ for sufficiently small $δ$ from $0 < c_p < 1$. Then it suffices to show that the remaining three possible market outcomes have the ordering specified in Lemma A.7 as $π_r$ increases, and we can define the upper bound on $R$ as the intersection of two curves.

Comparing the region of the parameter space in which the interior optimal prices of these cases induce their respective market outcomes, we find that there is an overlap between $0 < v_{nr} < v_r < 1$ and $0 < v_r < 1$. We can find the $π_r$ boundary between these two cases by equating their profits (A.58) and (A.65). For sufficiently low $δ$, the boundary between these two cases can be expressed as $R = A_0 + \sum_{k=1}^{∞} a_k δ^k$, where

$$A_0 = -6α^3 π^3_r + 2απ_r(9 + 3\sqrt{1 + απ_r + α^3π^3_r + α^4π^4_r} - 4\sqrt{5 + απ_r(-2 + 5απ_r) + 4\sqrt{1 + απ_r + α^3π^3_r + α^4π^4_r}}) - 2(-1 + \sqrt{1 + απ_r + α^3π^3_r + α^4π^4_r})(-3 + \sqrt{5 + απ_r(-2 + 5απ_r) + 4\sqrt{1 + απ_r + α^3π^3_r + α^4π^4_r}}) + α^2π^2_r(-9 + 2\sqrt{5 + απ_r(-2 + 5απ_r) + 4\sqrt{1 + απ_r + α^3π^3_r + α^4π^4_r}}) \times \left(2π_r(-1 + απ_r(4 - απ_r) + \sqrt{1 + απ_r + α^3π^3_r + α^4π^4_r}(-3 - 3απ_r + \sqrt{5 + απ_r(-2 + 5απ_r) + 4\sqrt{1 + απ_r + α^3π^3_r + α^4π^4_r}}) - 1 \right)^{-1} (A.76)$$

A.27
$A_0$ as a function of $\pi_r$ is strictly increasing for $\pi_r > 0$, so this means that (viewing the boundary as a function of $\pi_r$ so that $R = f(\pi_r)$ is the boundary between the two regions), the function is invertible for sufficiently small $\delta$. Define
\begin{align*}
\pi_0 & \triangleq f^{-1}(R) \quad (A.77)
\end{align*}
as the inverse of this $R$ boundary above. Note $\pi_0 > 0$ for $R > R_2$ since $0 < A_0|_{\pi_r=0} < R_2$ and $A_0$ is increasing in $\pi_r$. Then for $\pi_r < \pi_0$, the profit of $0 < v_{nr} < v_r < 1$ dominates the profit under $0 < v_r < 1$. The reverse is true when the condition is reversed. To help with the exposition in the paper, we define $\hat{\pi}$ as:
\begin{align*}
\hat{\pi} = \max(0, \pi_0). \quad (A.78)
\end{align*}
In particular, $\hat{\pi} = \pi_0$ under the conditions of this lemma.

There is also an overlap between the regions in which the interior optimal price for $0 < v_{nr} < v_r < v_p < 1$ induces that market outcome and in which the interior optimal price for $0 < v_r < 1$ induces that outcome. Equating the profits given in (A.62) and (A.65),
\begin{align*}
\pi_2 = \frac{c_p \alpha}{R^2 - c_p R \alpha} + \sum_{k=1}^{\infty} a_k \delta^k. \quad (A.79)
\end{align*}
Then the profit of $0 < v_r < 1$ dominates the profit under $0 < v_{nr} < v_r < v_p < 1$ if and only if $\pi_r < \pi_2$.

To complete the proof, we want to show that $\pi_2 > \hat{\pi}$. It suffices to show that sufficiently close to the boundary $R = R_2$, we have $\pi_2 < 1$ and $\Pi_1^* - \Pi_{II}^* > 0$ at $\pi_r = \pi_2$. That $\pi_2 < 1$ for sufficiently low $\delta$ follows from conditions on $c_p$ and $\alpha$ from the focal region assumptions.

Similarly, taking the difference between the profits and using the focal region assumptions, we have that $\Pi_1^* - \Pi_{II}^* > 0$ for $\pi_r$ sufficiently close to $\pi_r = \pi_2$. This implies that $\pi_2 > \hat{\pi}$ for $R$ sufficiently close to $R_2$. Moreover, since $\pi_2$ is a decreasing function of $R$ and $\pi_0$ is increasing in $R$, there is a unique intersection between these two curves. Define $\tilde{\omega}$ as
\begin{align*}
\tilde{\omega} = \arg_R (\pi_0(R) = \pi_2(R)). \quad (A.80)
\end{align*}
Note that $\tilde{\omega} > \frac{\alpha}{2 - c_p}$, since $\pi_0 < \pi_2$ for $R$ sufficiently close to $R_2$ (which has an asymptotic limit of $\frac{\alpha}{2 - c_p}$ for small $\delta$). Altogether, the above shows that there exist bounds $\tilde{\delta} > 0$ and $\tilde{\omega} \in (\frac{\alpha}{2 - c_p}, \tilde{\omega})$ such that if $\delta < \tilde{\delta}$ and $R \in (R_2, \tilde{\omega})$, then:
(a) if $0 \leq \pi_r < \hat{\pi}$, then the equilibrium outcome is $0 < v_{nr} < v_r < 1$;
(b) if $\hat{\pi} \leq \pi_r < \pi_2$, then the equilibrium outcome is $0 < v_r < 1$;
(c) if $\pi_2 \leq \pi_r \leq 1$, then the equilibrium outcome is $0 < v_{nr} < v_r < v_p < 1$,
where $\hat{\pi}$ and $\pi_2$ are characterized in (A.78) and (A.79) respectively. ■

A.28
Lemma A.8. There exist bounds \( \tilde{\delta} > 0 \) and \( \hat{\omega} > \frac{\alpha}{2-c_p} \) such that if \( \delta < \tilde{\delta} \), then:

(a) if \( 0 \leq R \leq \hat{R}_1 \), then the equilibrium outcome is \( 0 < v_r < 1 \) for any \( \pi_r \);

(b) if \( \hat{R}_1 < R \leq \hat{R}_2 \), then the equilibrium outcome is \( 0 < v_r < 1 \) for \( \pi_r \in [0, \pi_1) \), and the equilibrium outcome is \( 0 < v_r < v_p < 1 \) for \( \pi_r \in [\pi_1, 1] \);

(c) if \( \hat{R}_2 < R \leq R_2 \), then the equilibrium outcome is \( 0 < v_{nr} < v_r < 1 \) for \( \pi_r \in [0, \hat{\pi}) \), the equilibrium outcome is \( 0 < v_r < 1 \) for \( \pi_r \in [\hat{\pi}, \pi_1) \), and the equilibrium outcome is \( 0 < v_r < v_p < 1 \) for \( \pi_r \geq \pi_2 \);

(d) if \( R_2 < R \leq \hat{\omega} \), then the equilibrium outcome is \( 0 < v_{nr} < v_r < 1 \) for any \( \pi_r \in [0, \hat{\pi}) \), the equilibrium outcome is \( 0 < v_r < 1 \) for \( \pi_r \in [\hat{\pi}, \pi_2) \), and the equilibrium outcome is \( 0 < v_{nr} < v_r < v_p < 1 \) for \( \pi_r \geq \pi_2 \),

where the bounds on \( R \) and \( \pi_r \) are characterized in the proof below.

Proof of Lemma A.8: The proof of this lemma follows closely from the proofs of Lemmas A.6 and A.7.

Firstly, consider the boundary given in (A.70) marking the boundary between \( 0 < v_r < 1 \) and \( 0 < v_r < v_p < 1 \). Taking the derivative with respect to \( \pi_r \), we see that this boundary (when viewing \( R \) as a function of \( \pi_r \)) is decreasing in \( \pi_r \) for sufficiently low \( \pi_r \). Then consider the boundary between \( 0 < v_r < 1 \) and \( 0 < v_{nr} < v_r < 1 \) given in (A.79). Rewriting this boundary in terms of \( R \) as a function in terms of \( \pi_r \), we have

\[
R = \frac{c_p\alpha}{2} + \frac{\sqrt{c_p\alpha(4 + c_p\alpha\pi_r)}}{2\sqrt{\pi_r}} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{A.81}
\]

Taking the derivative of this with respect to \( \pi_r \), we again see that this boundary is decreasing in \( \pi_r \) for sufficiently small \( \delta \).

Then from the proof of Lemma A.7, the boundary between \( 0 < v_{nr} < v_r < 1 \) and \( 0 < v_r < 1 \) (given in (A.77)) is strictly increasing in \( \pi_r \) when that boundary is viewed in terms of \( R \) as a function of \( \pi_r \).

By the proof of Lemma A.6, we have that \( 0 < v_{nr} < v_p < 1 \) and \( 0 < v_{nr} < 1 \) cannot be induced in equilibrium for \( R \leq \hat{\omega} \), where \( \hat{\omega} \) is the same as in Lemma A.7. To complete the proof for all \( \pi_r \) when \( R \leq \hat{\omega} \), it suffices to examine the boundaries between the remaining regions.

Since the boundary in (A.70) marking the boundary between \( 0 < v_r < 1 \) and \( 0 < v_r < v_p < 1 \) is decreasing in \( \pi_r \), the smallest \( R \) value at this boundary is when \( \pi_r = 1 \). Evaluating (A.70), we define \( \hat{R}_1 \) as

\[
\hat{R}_1 = \frac{1}{(1-c_p)^2} - 1 + \sum_{k=1}^{\infty} a_k \delta^k. \tag{A.82}
\]

A.29
Similarly, since the boundary between $0 < v_{nr} < v_r < 1$ and $0 < v_r < 1$ (given in (A.77)) is strictly increasing in $\pi_r$, it follows that the smallest $R$ value at this boundary is when $\pi_r = 0$. Evaluating (A.76) at $\pi_r = 0$, we define $\hat{R}_2$ as

$$\hat{R}_2 = \frac{\alpha}{2} + \sum_{k=1}^{\infty} a_k \delta^k. \quad (A.83)$$

Note that $\hat{R}_1 < \hat{R}_2$ under the conditions of the focal region, with $0 < c_p < 2 - \sqrt{3}$ and $\frac{2}{(1-c_p)^2} - 2 < \alpha < 2(2 - c_p)^2$.

For $R_2$, this is the boundary between $0 < v_r < v_p < 1$ and $0 < v_{nr} < v_r < v_p < 1$, which is given by (A.71) in the proof of Lemma A.6. We write it below here for clarity.

$$R_2 = \frac{\alpha}{2 - c_p} - \left( \frac{\alpha}{2} + \frac{c_p^2}{\pi_r} \right) \delta + \sum_{k=2}^{\infty} a_k \delta^k. \quad (A.84)$$

Under the conditions of the focal region, $\hat{R}_2 < R_2$ for sufficiently low $\delta$. As shown in Lemma A.7, that $R_2 < \hat{\omega}$ follows from the boundary between $0 < v_r < 1$ and $0 < v_{nr} < v_r < v_p < 1$ being decreasing in $\pi_r$ (when viewing the boundary in terms of $R$ as a function of $\pi_r$).

Define $\pi^*$ as the solution of (A.70) in terms of $\pi_r$,

$$\pi^* = \frac{(2 - c_p)c_p}{(1-c_p)^2 R} + \sum_{k=1}^{\infty} a_k \delta^k. \quad (A.85)$$

The lemma then follows from the statements of Lemmas A.6 and A.7, with

$$\pi_1 = \min(\pi^*, 1), \quad (A.86)$$

$\hat{\pi}$ coming from (A.78), and $\pi_2$ coming from (A.79). That $\pi_1 > \hat{\pi}$ follows from $\pi_1 > \tilde{\pi}$ (given in (A.72)) for $R < R_2$ and $\pi_0 < \tilde{\pi}$ for $R < R_2$. $\blacksquare$

**Lemma A.9.** In the setting of ransomware spread through other vectors (non-patchable), there exists a bound $\tilde{\delta} > 0$ such that if $\delta < \tilde{\delta}$, then:

(a) if $0 \leq R \leq \tilde{R}_1$ (where $\tilde{R}_1$ is defined in the proof below), then the equilibrium consumer market structure is $0 < v_r < 1$;

(b) if $\tilde{R}_1 < R < \alpha(1 - \delta)$, then the equilibrium consumer market structure is $0 < v_{nr} < v_r < 1$;

(c) if $R \geq \alpha(1 - \delta)$, then the equilibrium consumer market structure is $0 < v_{nr} < 1$.

**Proof of Lemma A.9:** From Lemma A.2, we have the consumer market equilibrium outcomes across the parameter space, given a price $p$. We use this to specify the interior optimal price and vendor’s profit at that interior optimal price for each of market outcomes.
Given a price $p$, the region of the parameter space defining $0 < v_{nr} < 1$ is given in part (I) of Lemma A.1. For this case, we have

$$v_{nr} = \frac{-1 + \pi_r \alpha + \sqrt{1 + \pi_r \alpha (-2 + 4p + \pi_r \alpha)} - 1}{2\pi_r \alpha}. \quad (A.87)$$

The profit function in this case is $\Pi_I(p) = p(1 - v_{nr}(p))$. Let $C_I$ be the compact closure of the region of the parameter space defining $0 < v_{nr} < 1$, given in part (I) of Lemma A.2. By the Weierstrass extreme value theorem, there exists $p$ in $C_I$ that maximizes $\Pi_I(p)$. If this $p$ is interior to $C_I$, the unconstrained maximizer satisfies the first-order condition. The Weierstrass extreme value applies for all regions, and we will not state this for other regions.

Differentiating the profit function with respect to $p$ and solving for the positive root of the quadratic, we have that

$$p^*_I = \frac{1}{9} \left(4 - \frac{1}{\pi_r \alpha} - \pi_r \alpha + \frac{\sqrt{1 + \pi_r \alpha (\pi_r \alpha)^3 + (\pi_r \alpha)^4}}{\pi_r \alpha}\right). \quad (A.88)$$

Given a price $p$, the region of the parameter space defining $0 < v_{nr} < v_r < 1$ is given in Lemma A.2. This also a special case of part (III) of Lemma A.1. For this case, we have

$$v_{nr} = \frac{-1 + \pi_r \alpha + \sqrt{1 + \pi_r \alpha (-2 + 4p + \pi_r \alpha)} - 1}{2\pi_r \alpha}. \quad (A.89)$$

The profit function in this case is $\Pi_{III}(p) = p(1 - v_{nr}(p))$. Let $C_{III}$ be the compact closure of the region of the parameter space defining $0 < v_{nr} < v_r < 1$, given in Lemma A.2. As in the previous case, there exists $p$ in $C_{III}$ that maximizes $\Pi_{III}(p)$. If this $p$ is interior to $C_{III}$, the unconstrained maximizer satisfies the first-order condition.

Differentiating the profit function with respect to $p$ and solving for the positive root of the quadratic, we have that

$$p^*_{III} = \frac{1}{9} \left(4 - \frac{1}{\pi_r \alpha} - \pi_r \alpha + \frac{\sqrt{1 + \pi_r \alpha (\pi_r \alpha)^3 + (\pi_r \alpha)^4}}{\pi_r \alpha}\right). \quad (A.90)$$

The profit corresponding to this price for this case is given by:

$$\Pi^*_{III} = \frac{3 + 3\alpha \pi_r - \sqrt{5 + \pi_r \alpha (-2 + 5\pi_r \alpha) + 4\sqrt{1 + \pi_r \alpha (\pi_r \alpha)^3 + (\pi_r \alpha)^4}}}{54(\pi_r \alpha)^2} \times \left(-1 + \pi_r \alpha (4 - \pi_r \alpha) + \sqrt{1 + \pi_r \alpha (\pi_r \alpha)^3 + (\pi_r \alpha)^4}\right). \quad (A.91)$$

Given a price $p$, the region of the parameter space defining $0 < v_r < 1$ is given in Lemma A.2, which is a special case of part (V) of Lemma A.1.
From (A.28), we have that
\[ v_r = \frac{-1 - R\pi_r + \delta\pi_r\alpha + \sqrt{4\delta\pi_r\alpha(p + R\pi_r) + (1 + R\pi_r - \delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}. \] (A.92)

The vendor’s profit function is given as \( \Pi_V(p) = p(1 - v_r(p)) \). Solving for FOC gives
\[
p^*_V = \left( -1 - 2R\pi_r + 4\delta\pi_r\alpha - R^2\pi_r^2 - 2R\delta\alpha\pi_r^2 - (\delta\pi_r\alpha)^2 + (1 + R\pi_r + \delta\pi_r\alpha)\sqrt{1 + \pi_r(2R - \delta\alpha + (R + \delta\alpha)^2\pi_r)} \right) \left( 9\delta\pi_r\alpha \right)^{-1}. \] (A.93)

Substituting (A.93) into the profit function of this case yields the associated maximal profit of this case. Characterizing this profit expression in terms of a Taylor Series expansion,
\[
\Pi^*_V = \frac{1}{4(1 + R\pi_r)} + \sum_{k=1}^{\infty} a_k \delta^k. \] (A.94)

Now that we have found the interior optimal prices for these regions, we use Lemma A.2 to find conditions under which the interior optimal price for a case lies within the set of conditions defining that case. Omitting the algebra, there is an overlap between the regions in which \( 0 < v_{nr} < v_r < 1 \) and \( 0 < v_r < 1 \). There is no overlap between \( 0 < v_{nr} < 1 \) and any other region. However, \( 0 < v_{nr} < 1 \) and \( 0 < v_{nr} < v_r < 1 \) share a boundary \( R = \alpha(1 - \delta) \) such that if \( R \geq \alpha(1 - \delta) \), then \( 0 < v_{nr} < 1 \) is induced under optimal pricing.

For \( R < \alpha(1 - \delta) \), either \( 0 < v_{nr} < v_r < 1 \) or \( 0 < v_r < 1 \) will be the induced market outcome. Comparing the region of the parameter space in which the interior optimal prices of these cases induce their respective market outcomes, we find the overlap between \( 0 < v_{nr} < v_r < 1 \) and \( 0 < v_r < 1 \). We can find the \( R \) boundary (or equivalently, the \( \pi_r \) boundary) between these two cases by equating their profits, (A.91) and (A.94). For sufficiently low \( \delta \), the boundary between these two cases can be expressed as
\[
\tilde{R}_1 = A_0 + \sum_{k=1}^{\infty} a_k \delta^k, \text{ where } \] (A.95)
\[ A_0 = \left( -6\alpha^3\pi_r^3 + 2\alpha\pi_r(9 + 3\sqrt{1 + \alpha\pi_r + \alpha^3\pi_r^3 + \alpha^4\pi_r^4} - 4\sqrt{5 + \alpha\pi_r(-2 + 5\alpha\pi_r) + 4\sqrt{1 + \alpha\pi_r + \alpha^3\pi_r^3 + \alpha^4\pi_r^4}) - 2(-1 + \sqrt{1 + \alpha\pi_r + \alpha^3\pi_r^3 + \alpha^4\pi_r^4})(-3 + \sqrt{5 + \alpha\pi_r(-2 + 5\alpha\pi_r) + 4\sqrt{1 + \alpha\pi_r + \alpha^3\pi_r^3 + \alpha^4\pi_r^4}}) + \alpha^2\pi_r^2(-9 + 2\sqrt{5 + \alpha\pi_r(-2 + 5\alpha\pi_r) + 4\sqrt{1 + \alpha\pi_r + \alpha^3\pi_r^3 + \alpha^4\pi_r^4}})(-3 + 3\alpha\pi_r + \sqrt{5 + \alpha\pi_r(-2 + 5\alpha\pi_r) + 4\sqrt{1 + \alpha\pi_r + \alpha^3\pi_r^3 + \alpha^4\pi_r^4}}) \right)^{-1} \quad \text{(A.96)} \]

\( A_0 \) as a function of \( \pi_r \) is strictly increasing for \( \pi_r > 0 \), and since \( A_0 < \alpha \) both when \( \pi_r = 0 \) and when \( \pi_r = 1 \), it follows that \( A_0 < \alpha \) for all \( \pi_r \in [0, 1] \). Then \( \tilde{R}_1 < \alpha(1 - \delta) \) for sufficiently small \( \delta \). Then for \( R > \tilde{R}_1 \), the profit of \( 0 < v_{nr} < v_r < 1 \) dominates the profit under \( 0 < v_r < 1 \). The reverse is true when the condition is reversed, and the vendor is indifferent between these cases at \( R = \tilde{R}_1 \). ■

**Lemma A.10.** In the setting of non-patchable ransomware, there exists a bound \( \tilde{\delta} > 0 \) such that if \( \tilde{R}_2 < R < \tilde{R}_3 \):

(a) if \( 0 \leq \pi_r < \pi' \) (where \( \hat{\pi} \) is defined in the proof below), then the equilibrium consumer market structure is \( 0 < v_{nr} < v_r < 1 \);

(b) if \( \pi' \leq \pi_r \leq 1 \), then the equilibrium consumer market structure is \( 0 < v_r < 1 \),

where \( \tilde{R}_2 \) and \( \tilde{R}_3 \) are defined in the proof below.

**Proof of Lemma A.10:** This follows directly from Lemma A.9, viewing the characterization in terms of \( \pi_r \) instead of \( R \) and focusing on a range of \( R \) values above \( \frac{1}{2} \alpha(1 - \delta) \). Specifically, define

\[ \tilde{R}_2 = \tilde{R}_1|_{\pi_r=0} = \frac{1}{2} \alpha(1 - \delta) \quad \text{(A.97)} \]

as the value of \( \tilde{R}_1 \) when \( \pi_r = 0 \), and define

\[ \tilde{R}_3 = \tilde{R}_1|_{\pi_r=1} \quad \text{(A.98)} \]

as the value of \( \tilde{R}_1 \) when \( \pi_r = 1 \). Also, since \( A_0 \) is strictly increasing in \( \pi_r \), this means that (viewing the boundary as a function of \( \pi_r \)), the function is invertible for sufficiently small \( \delta \). Define

\[ \pi' \triangleq \max(0, \tilde{R}_1^{-1}(\pi_r)) \quad \text{(A.99)} \]

as the max of 0 and the inverse function \( \tilde{R}_1(\pi_r) \). Then the profit of \( 0 < v_{nr} < v_r < 1 \) dominates the profit under \( 0 < v_r < 1 \) iff \( \pi_r < \pi' \) (equivalent to \( R > \tilde{R}_1 \) in Lemma A.9). ■
A.4 Summary of Notation

To assist the reader, we have included a table summarizing the notation of boundaries between regions defined in the lemmas of Section A.3 and used throughout the paper.

<table>
<thead>
<tr>
<th>Result</th>
<th>Sensitivity analysis in parameter</th>
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<td>Proposition 1</td>
<td>$R$</td>
<td>$R_1$</td>
<td>boundary between equilibrium regions (A) and (B) in Figure 1 (between $0 &lt; v_r &lt; 1$ and $0 &lt; v_r &lt; v_p &lt; 1$ regions)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_2$</td>
<td>boundary between equilibrium regions (B) and (C) in Figure 1 (between $0 &lt; v_r &lt; v_p &lt; 1$ and $0 &lt; v_{nr} &lt; v_r &lt; v_p &lt; 1$ regions)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_3$</td>
<td>boundary between equilibrium regions (C) and (D) in Figure 1 (between $0 &lt; v_{nr} &lt; v_r &lt; v_p &lt; 1$ and $0 &lt; v_{nr} &lt; v_p &lt; 1$ regions)</td>
</tr>
<tr>
<td>Propositions 2 and 3</td>
<td>$\pi_r$</td>
<td>$\pi_1$</td>
<td>boundary between equilibrium regions (A) and (B) in Figure 1 (between $0 &lt; v_r &lt; 1$ and $0 &lt; v_r &lt; v_p &lt; 1$ regions)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pi_2$</td>
<td>boundary between equilibrium regions (A) and (C) in Figure 1 (between $0 &lt; v_r &lt; 1$ and $0 &lt; v_{nr} &lt; v_r &lt; v_p &lt; 1$ regions)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\hat{\pi}$</td>
<td>boundary between equilibrium regions (E) and (A) in Figure 1 (between $0 &lt; v_{nr} &lt; v_r &lt; 1$ and $0 &lt; v_r &lt; 1$ regions)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\tilde{\pi}$</td>
<td>$\tilde{\pi} = min(\pi_1, \pi_2)$</td>
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<td>Proposition 6</td>
<td>$R$</td>
<td>$\tilde{R}_1$</td>
<td>boundary between equilibrium regions (A) and (E) in Figure 5 (between $0 &lt; v_r &lt; 1$ and $0 &lt; v_{nr} &lt; v_r &lt; 1$ regions)</td>
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<tr>
<td>Propositions 7 and 8</td>
<td>$\pi_r$</td>
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<td>boundary between equilibrium regions (E) and (A) in Figure 5 (between $0 &lt; v_{nr} &lt; v_r &lt; 1$ and $0 &lt; v_r &lt; 1$ regions)</td>
</tr>
</tbody>
</table>

Table A.1: Description of bounds used in Propositions 1, 2, 3, 6, 7, and 8.
A.5 Proofs of Propositions

Proof of Proposition 1: From Lemma A.6, we have how the equilibrium market outcome changes in $R$. To complete the proof, we do comparative statics on the vendor’s equilibrium price, profits, and market size with respect to $R$ for each of the market outcomes.

For $R < R_1$, the equilibrium market outcome is $0 < v_r < 1$. The vendor’s equilibrium price for sufficiently small $\delta$ has a Taylor series expansion of the form

$$p^*_V = \frac{1}{2} - \frac{\alpha \pi_r}{8(1 + R \pi_r)^2} \delta + \sum_{k=2}^{\infty} a_k \delta^k.$$  \hfill (A.100)

From (A.62), we have the an asymptotic expression in $\delta$ of the vendor’s equilibrium profit. Finally, the market size of this case is $1 - v^*_r$, and this has an asymptotic expansion given by

$$M^*_V = \frac{1}{2(1 + R \pi_r)} + \sum_{k=1}^{\infty} a_k \delta^k.$$  \hfill (A.101)

Taking the derivatives of $p^*_V$, $\Pi^*_V$, and $M^*_V$ with respect to $R$, we have that for sufficiently low $\delta$, the vendor’s price in this case increases in $R$ while the vendor’s profit and market size decrease.

For $R$ in $R_1 \leq R < R_2$, the equilibrium market outcome is $0 < v_r < v_p < 1$. The vendor’s equilibrium price in this case is given by (A.68), and the vendor’s profit at this price is given by (A.69). The market size of this case is given by $1 - v^*_r$, and this has an asymptotic expansion given by

$$M^*_{VI} = \frac{1 - c_p}{2} + \frac{c_p \alpha^2}{2R^2 \pi_r} \delta + \sum_{k=2}^{\infty} a_k \delta^k.$$  \hfill (A.102)

Taking the derivatives of $p^*_{VI}$, $\Pi^*_{VI}$, and $M^*_{VI}$ with respect to $R$, we have that for sufficiently low $\delta$, the vendor’s price, profit, and market size all decrease in $R$.

For $R$ in $R_2 \leq R < R_3$, the equilibrium market outcome is $0 < v_{nr} < v_r < v_p < 1$. The vendor’s equilibrium price in this case is given by (A.61), and the vendor’s profit at this price is given by (A.62). The market size of this case is given by $1 - v^*_{nr}$, and this has an asymptotic expansion given by

$$M^*_{IV} = \frac{1}{2} + \frac{c_p \alpha^2}{8(R(R - c_p \alpha))} \delta + \sum_{k=2}^{\infty} a_k \delta^k.$$  \hfill (A.103)

Taking the derivatives of $p^*_{IV}$, $\Pi^*_{IV}$, and $M^*_{IV}$ with respect to $R$, we have that for sufficiently low $\delta$, the vendor’s price, profit, and market size all increase in $R$.

For $R \in [R_3, \omega)$, the equilibrium market outcome is $0 < v_{nr} < v_p < 1$, and the comparative statics with respect to $R$ are still driven by movement of $p_{\text{boundary}}$ in (A.73). In that region of $R$, the vendor’s price at the boundary increases in $R$, and the vendor’s profit increases in
As $\phi$ moves toward the interior optimal $p^*_\phi$ of this case (which does not change in $R$) instead of being constrained by a boundary condition with $0 < v_{nr} < v_r < 1$. In the proof of Lemma A.4, we showed that $p^*_\phi(v_{nr})$ is a strictly increasing function of $v_{nr}$. Since the market size is $1 - v_{nr}$, it follows that the market size shrinks as $R$ increases.

For $R \geq \omega$, the consumer market equilibrium is again $0 < v_{nr} < v_p < 1$, and the vendor can achieve this using his interior optimal price. At this point, the pricing is no longer driven by boundary pricing, and the vendor’s price, market size, and profits remain constant in $R$.

This completes the proof of Proposition 1. ■

Proof of Proposition 2: From Lemma A.8, we have how the equilibrium market outcome changes in $\pi_r$ for all $R \leq \hat{\omega}$. To complete the proof, we do comparative statics on the vendor’s equilibrium price with respect to $\pi_r$ for each of the market outcomes and compare the price values at the $\pi_r$ values marking the regime switches.

For $R \leq \hat{R}_1$, the equilibrium outcome is $0 < v_r < 1$. The vendor’s price of this case is given in (A.64). This has an asymptotic expression in $\delta$ given by

$$p^*_V = \frac{1}{2} - \frac{\pi_r \alpha}{8(1 + R\pi_r)^2} \delta + \sum_{k=2}^{\infty} a_k \delta^k. \quad (A.104)$$

In this low range of $R$, there is no strategic price jump or drop to induce another market outcome, and taking the derivative with respect to $\pi_r$, we have that $\frac{d}{d \pi_r}[p^*_V] < 0$ for all $\pi_r \in (0, \frac{1}{R})$. Note that $\hat{R}_1 < 1$ for sufficiently low $\delta$ within the focal region assumptions on $c_p$, so this means that the price is decreasing in $\pi_r$ for all $\pi_r < 1$ in this region of $R$.

For $\hat{R}_1 < R \leq \hat{R}_2$, the equilibrium outcome is $0 < v_r < 1$ for $\pi_r < \pi_1$ and $0 < v_r < v_p < 1$ for $\pi_r > \pi_1$. Again, taking the derivative with respect to $p^*_V$ shows that the price is decreasing for all $\pi_r < \pi_1$ in this region of $R$. The price of $0 < v_r < v_p < 1$ is given in (A.68). Taking the derivative of this with respect to $\pi_r$ shows that this price is strictly decreasing for all $\pi_r$, regardless of $R$. Moreover, note that the price given in (A.68) is arbitrarily close to $\frac{1-c_p}{2}$ for sufficiently small $\delta$ while (A.104) is strictly larger than that. Hence, there is a price drop at $\pi_r = \pi_1$.

For $\hat{R}_2 < R \leq \hat{\omega}$, there are two cases depending on whether $R > \hat{R}_2$ from Lemma A.8. In either case, for $\pi_r < \hat{\pi}$, the consumer market equilibrium structure that arises under optimal pricing is $0 < v_{nr} < v_r < 1$. For $\hat{R}_2 < R \leq \hat{R}_2$, then if $\pi_r \in (\hat{\pi}, \pi_1)$, the equilibrium outcome is $0 < v_r < 1$. For $\hat{R}_2 < R \leq \hat{\omega}$, then if $\pi_r \in (\hat{\pi}, \pi_2)$, the equilibrium outcome is $0 < v_r < 1$. Furthermore, note that when $\hat{R}_2 < R \leq \hat{R}_2$, then $\pi_1 \leq \pi_2$ under the conditions of the focal region. On the other hand, when $\hat{R}_2 < R \leq \hat{\omega}$, we can define $\hat{\pi} = \min(\pi_1, \pi_2)$ as the $\pi_r$ cutoff above which $0 < v_r < 1$ no longer holds in equilibrium.

The vendor’s price, provided in (A.57), is again given as
\[ p_{III}^* = \frac{1}{9} \left( 4 - \frac{1}{\pi_r \alpha} - \pi_r \alpha + \sqrt{1 + \pi_r \alpha + (\pi_r \alpha)^3 + (\pi_r \alpha)^4} \right). \] (A.105)

Note that \( \lim_{\pi_r \to 0} [p_{III}^*] = \frac{1}{2} \). Furthermore, \( \frac{d}{d\pi_r} [p_{III}^*] < 0 \) under the conditions of the proposition. Therefore, \( p_{III}^* < \frac{1}{2} \) for \( \pi_r > 0 \).

On the other hand, when \( \pi_r \in (\hat{\pi}, \tilde{\pi}) \), the consumer market equilibrium structure that arises under optimal pricing is \( 0 < v_r < 1 \).

The asymptotic expression for the vendor’s price, provided in (A.104). Taking the derivative with respect to \( \pi_r \), we have that \( \frac{d}{d\pi_r} [p_V^*] < 0 \) for sufficiently small \( \delta \) is equivalent to \( R\pi_r < 1 \). Using that \( R \) is close to \( R = \frac{\alpha}{2-c_p} \) for sufficiently small \( \delta \) and using \( c_p < \frac{1}{2} (2 - \sqrt{2}) \) which is implied by the conditions on \( c_p \) from the focal region assumptions, we have that \( R\pi_r < 1 \) so that the price is decreasing in \( \pi_r \) over this region. Moreover, for sufficiently small \( \delta \), \( p_V^* \) is close to \( \frac{1}{2} \). This implies there is a price hike at \( \pi_r = \hat{\pi} \).

Lastly, when \( \pi_r > \min(\pi_1, \pi_2) \), the equilibrium outcome is either \( 0 < v_r < v_p < 1 \) if \( R < R_2 \) or \( 0 < v_{nr} < v_r < v_p < 1 \) if \( R \geq R_2 \). In the former case, the comparative statics and price drop at \( \pi_r = \pi_1 \) are the same as before when \( \hat{R}_1 < R < \hat{R}_2 \). The vendor’s price in the latter case, provided in (A.61), is given below:

\[ p_{IV}^* = \frac{R - c_p \alpha}{2R} + \frac{c_p \alpha^2(4c_p + 3R \pi_r)\delta}{8R^3 \pi_r} + \sum_{k=2}^{\infty} a_k \delta^k. \] (A.106)

Taking the derivative with respect to \( \pi_r \), we have \( \frac{d}{d\pi_r} [p_{IV}^*] < 0 \) for sufficiently small \( \delta \). Moreover, \( \frac{R-c_p \alpha}{2R} < \frac{1}{2} \) for \( c_p > 0, \alpha > 0 \), and \( R > 0 \) so that for sufficiently small \( \delta \), \( p_{IV}^* < p_V^* \) (i.e., there is a price drop at \( \pi_r = \pi_2 \)). \( \blacksquare \)
**Full Statement of Proposition 3:** Suppose the conditions of Proposition 2 are satisfied.

(a) The vendor’s profit and market size are piecewise decreasing in $\pi_r$.

(b) If $0 < R < R_2$:

(i) if $0 \leq R \leq \hat{R}_1$, the size of the ransom-paying population decreases in $\pi_r$ and the expected total ransom paid increases in $\pi_r$ for all $\pi_r \in (0, 1)$.

(ii) if $\hat{R}_1 < R \leq \hat{R}_2$, the size of the ransom-paying population decreases in $\pi_r$ and the expected total ransom paid increases in $\pi_r$ for $\pi_r \in (0, \hat{\pi}_1)$. On the other hand, for $\pi_r \in (\hat{\pi}_1, 1)$, both the ransom-paying population size and the expected total ransom paid increase in $\pi_r$.

(iii) If $\hat{R}_2 < R \leq R_2$, the size of the ransom-paying population is constant in $\pi_r$ and the expected total ransom paid increases in $\pi_r$ for $\pi_r \in (0, \hat{\pi})$. For $\pi_r \in (\hat{\pi}, \pi_2)$, the ransom-paying population size decreases and the expected total ransom paid increases in $\pi_r$. For $\pi_r \in (\pi_2, 1)$, both the ransom-paying population size and the expected total ransom paid increase in $\pi_r$.

(c) If $R_2 \leq R < \hat{\omega}$, then the size of the ransom-paying segment is constant in $\pi_r$ and the expected total ransom paid is increasing in $\pi_r$ over $\pi_r \in (0, \hat{\pi})$. If $\pi_r \in (\hat{\pi}, \pi_2)$, then the size of the ransom-paying segment is decreasing in $\pi_r$, and the expected total ransom paid is increasing in $\pi_r$. If $\pi_r \in (\pi_2, 1)$, then both the size of the ransom-paying segment and expected total ransom paid are decreasing in $\pi_r$.

where $\hat{R}_1 = \frac{1}{(1 - c_p)^2} - 1 + \bar{\kappa}_1(\delta)$, $\hat{R}_2 = \frac{\alpha}{2} + \bar{\kappa}_2(\delta)$, and $R_2 = \frac{\alpha}{2 - c_p} + \bar{\kappa}_3(\delta)$. The vendor’s profit and market size are decreasing in $\pi_r$ on each of the specified intervals above. Finally, when $R_2 < R \leq \hat{\omega}$, there exists an open interval centered at $\hat{\pi}$ such that the expected total ransom paid is discontinuously higher in the upper half of the interval in comparison to its measure in the lower half.

**Proof of Full Statement of Proposition 3:** From Lemma A.8, we have how the equilibrium market outcome changes in $\pi_r$. To complete the proof, we do comparative statics on the vendor’s equilibrium profit, market size, the ransom-paying population size, and the expected total ransom paid with respect to $\pi_r$ for each of the market outcomes. We then compare the values for the ransom-paying population size and expected total ransom paid to the left and right of the boundary $\pi_r = \pi_2$ marking the regime switch between $0 < v_r < 1$ and $0 < v_{nr} < v_r < v_p < 1$.

Consider $0 \leq R \leq \hat{R}_1$. By Lemma A.8, the equilibrium outcome is $0 < v_r < 1$. For $0 < v_r < 1$, the size of the consumer segment willing to pay ransom in equilibrium is given as $r(\sigma^*) \triangleq 1 - v_r$. In this market structure, $v_r$ was given by (A.63). We provide it again below.
\[ v_r = -1 - R\pi_r + \delta\pi_r\alpha + \sqrt{4\delta\pi_r\alpha(p + R\pi_r) + (1 + R\pi_r - \delta\pi_r\alpha)^2} \div 2\delta\pi_r\alpha. \] (A.107)

The vendor's price \( p^*_V \) for sufficiently small \( \delta \) was given by (A.64).

Again, \( p^*_V \) has an asymptotic expression in \( \delta \) given by

\[ p^*_V = \frac{1}{2} - \frac{\pi_r\alpha}{8(1 + R\pi_r)^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \] (A.108)

Substituting this into the expression for \( v_r \) and simplifying \( r(\sigma^*) \), we have the equilibrium size of the consumer segment willing to pay ransom is

\[ r_V(\sigma^*) = \frac{1}{2(1 + R\pi_r)} + \sum_{k=1}^{\infty} a_k\delta^k. \] (A.109)

This is strictly decreasing in \( \pi_r \) for sufficiently small \( \delta \) for \( R \leq \hat{R}_1 \). Since \( 1 - v_r \) is also the market size of this case, this implies that the market size is decreasing in \( \pi_r \) in this case.

The vendor’s profit has an asymptotic expansion given in (A.65). For sufficiently small \( \delta \), this is decreasing in \( \pi_r \).

The expected total ransom paid in the case of \( 0 < v_r < 1 \) is given as

\[ T_V(\sigma^*) = \frac{R\pi_r}{4(1 + R\pi_r)^2} + \sum_{k=1}^{\infty} a_k\delta^k. \] (A.110)

That this is increasing in \( \pi_r \) for sufficiently small \( \delta \) is equivalent to \( R\pi_r < 1 \). Since \( R \leq \hat{R}_1 \) and \( \hat{R}_1 < 1 \) under the \( c_p \) conditions of the focal region, it follows that \( R\pi_r < 1 \) so that the expected total ransom paid is increasing in \( \pi_r \) in this range of \( R \).

For \( \hat{R}_1 < R \leq \hat{R}_2 \), the equilibrium outcome is \( 0 < v_r < 1 \) for \( \pi_r < \pi_1 \) and \( 0 < v_r < v_p < 1 \) for \( \pi_r \geq \pi_1 \). Note that \( R\pi_r < 1 \) still holds for \( R < \hat{R}_2 = \frac{\alpha}{2} \) and \( \pi_r < \pi_1 \) under the conditions of the focal region. Hence, the expected total ransom paid is still increasing in \( \pi_r \) in this region. Moreover, the comparative statics on the vendor’s profit, market size, and ransom-paying population remain the same as it was in the lower \( R \) region.

For \( \pi_r \geq \pi_1 \), the vendor’s profit is given in (A.69). Taking the derivative with respect to \( \pi_r \), we have that for sufficiently low \( \delta \), the vendor’s profit is decreasing in \( \pi_r \), and the sign of this direction does not change with the magnitude of \( R \).

The market size can be found by plugging in the vendor’s equilibrium price, given in (A.68), into the expression for \( v_r \), given in (A.67). This gives the expression for \( v_r \) in equilibrium under optimal pricing, and to find the market size, one would just subtract this from 1. The asymptotic expression for the market size in this case is given below.


\[ M_{VI}(\sigma^*) = \frac{1 - c_p}{2} + \frac{c_p^2 \alpha}{2R^2 \pi_r} \delta + \sum_{k=2}^{\infty} a_k \delta^k. \]  

(A.111)

Taking the derivative with respect to \( \pi_r \), we have that for sufficiently low \( \delta \), the equilibrium market size is decreasing in \( \pi_r \), and the sign of this direction does not change with the magnitude of \( R \).

Similarly, the equilibrium ransom-paying population size is \( v_p - v_r \) at the vendor’s optimal price. The asymptotic expression for this is given below.

\[ r_{VI}(\sigma^*) = \frac{2c_p}{R(1 + c_p)} - \frac{c_p \alpha}{R^3} \left( R + \frac{2c_p}{(1 + c_p)^2 \pi_r} \right) \delta + \sum_{k=2}^{\infty} a_k \delta^k. \]  

(A.112)

Taking the derivative with respect to \( \pi_r \), we have that for sufficiently low \( \delta \), the equilibrium ransom-paying population size is increasing in \( \pi_r \), and the sign of this direction does not change with the magnitude of \( R \).

The expected total ransom paid in this case is given by \( \pi_r u(\sigma)r(\sigma)R = \pi_r(v_p - v_r)^2R \). The asymptotic expression for this is given below.

\[ T_{VI}(\sigma^*) = \frac{4c_p^2 \pi_r}{(1 + c_p)^2 R^2} + \sum_{k=1}^{\infty} a_k \delta^k. \]  

(A.113)

Taking the derivative with respect to \( \pi_r \), we have that for sufficiently low \( \delta \), the equilibrium expected total ransom paid is increasing in \( \pi_r \), and the sign of this direction does not change with the magnitude of \( R \).

For \( 0 < v_{nr} < v_r < 1 \), the vendor’s equilibrium profit is given in (A.58). This is strictly decreasing under the conditions of the focal region. The market size is \( M_{III} = 1 - v_{nr} \) in this case, and the equilibrium \( v_{nr} \) can be found by plugging the vendor’s optimal price (A.57) into the expression defining \( v_{nr} \) in this case (A.56). Taking the derivative with respect to \( \pi_r \), the conditions of the focal region imply that the market size decreases in \( \pi_r \) in this case. The size of the consumer segment willing to pay ransom in equilibrium is given as \( r(\sigma^*) \triangleq 1 - v_r \). In this market structure, \( v_r = \frac{R}{\pi(1 - \delta)} \). This is constant in \( \pi_r \), and so the ransom-paying population size \( r_{III}(\sigma^*) = 1 - v_r \) is constant in \( \pi_r \). For the expected total ransom paid, that is given by \( T(\sigma^*) \triangleq \pi_r u(\sigma^*)r(\sigma^*)R \), where \( u(\sigma^*) \) is the size of the consumer segment willing to remain unpatched. The expected total ransom paid is \( T_{III}(\sigma^*) = \pi_r u(\sigma^*)r(\sigma^*)R \). For \( 0 < v_{nr} < v_r < 1 \), \( u(\sigma^*) = 1 - v_{nr} \) in equilibrium, where the equilibrium \( v_{nr} \) can be found by substituting the vendor’s optimal price (A.57) into the expression defining \( v_{nr} \) in (A.56). Under the conditions of the focal region, the expected total ransom paid is decreasing in \( \pi_r \).

Next, consider \( R_2 < R \leq R_2 \). By Lemma A.8, for \( \pi_r < \hat{\pi} \), the equilibrium market outcome is \( 0 < v_{nr} < v_r < 1 \). For \( \pi_r \in (\hat{\pi}, \pi_1) \), the equilibrium market outcome is \( 0 < v_r < 1 \). For \( \pi_r > \pi_1 \), the equilibrium outcome is \( 0 < v_r < v_p < 1 \).

For \( 0 < v_{nr} < v_r < 1 \), the vendor’s profit at the optimal price is given in (A.58). Taking the derivative of this with respect to \( \pi_r \) shows that this is strictly decreasing in \( \pi_r \) for all \( \pi_r \),
regardless of \( R \) or any other parameters.

The market size of this case is given as \( 1 - v_{nr} \). The expression for \( v_{nr} \) of this case is given in (A.56), and the vendor’s optimal price is given in (A.57). At the vendor’s optimal price, the expression for the equilibrium market size is decreasing in \( \pi_r \) for all \( \pi_r \), regardless of the values of the other parameters.

The size of the ransom-paying population is \( 1 - v_r \), where \( v_r = \frac{R}{\alpha(1 - \delta)} \). This is constant in \( \pi_r \).

The expected total ransom paid is \( \pi_r u(\sigma) r(\sigma) R \). Here, \( u(\sigma) = 1 - v_{nr} \) and \( r(\sigma) = 1 - v_r \), where \( v_r = \frac{R}{\alpha(1 - \delta)} \). Evaluating these at the vendor’s optimal price in (A.57) and taking the derivative with respect to \( \pi_r \), the expected total ransom paid increases in \( \pi_r \) for all \( \pi_r \), regardless of the values of the other parameters.

Next, for \( 0 < v_r < 1 \), again the comparative statics hold for the vendor’s profit, market size, and equilibrium ransom-paying population size since those comparative statics results did not depend on the magnitude of \( R \) for sufficiently small \( \delta \). We also note that \( R\pi_r < 1 \) still holds for \( \pi_r < \pi_1 \) and \( R < R_2 < \frac{\alpha}{2 - c_p} \).

For \( 0 < v_r < v_p < 1 \), the comparative statics still hold in the same way that they had for \( \hat{R}_1 < R < \hat{R}_2 \), since those did not depend on the value of \( R \) or any of the other parameters.

Lastly, for \( R_2 < R < \hat{\omega} \), the equilibrium outcome is \( 0 < v_{nr} < v_r < 1 \) for \( \pi_r < \hat{\pi} \), \( 0 < v_r < 1 \) for \( \pi_r \in (\hat{\pi}, \pi_2) \), and \( 0 < v_{nr} < v_r < v_p < 1 \) for \( \pi_r \in (\pi_2, 1) \).

For \( 0 < v_{nr} < v_r < 1 \), the comparative statics remains the same as for \( R < R_2 \), since those results did not depend on the magnitude of \( R \), \( \pi_r \), or any other parameters.

For \( 0 < v_r < 1 \), again the comparative statics results do not change for \( R \) close to \( R_2 \). Focusing on the region of \( R \) close to \( R = R_2 \), we have that the expected total ransom paid increases in \( R \) as long as \( c_p < 1 - \frac{1}{\sqrt{2}} \), which holds under the focal region.

Lastly, for \( 0 < v_{nr} < v_r < v_p < 1 \), the vendor’s profit was given in (A.62). This decreases in \( \pi_r \) under the assumptions of the focal region.

The market size of this case is \( M_{IV} = 1 - v_{nr} \). The equilibrium \( v_{nr} \) can be found by substituting the vendor’s optimal price (A.61) into the expression defining \( v_{nr} \) for this case in (A.60). The asymptotic expression for the market size of this case is given below.

\[
M_{IV}(\sigma^*) = \frac{1}{2} - \frac{c_p \alpha^2}{8R^2 - 8c_p R\alpha} \delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{A.114}
\]

Taking the derivative with respect to \( \pi_r \), the market size shrinks under the conditions of the focal region for \( R \) close to the boundary \( R = \frac{\alpha}{2 - c_p} \) for sufficiently small \( \delta \).

The size of the consumer segment willing to pay ransom in equilibrium is given as \( r(\sigma^*) \triangleq v_p - v_r \). In this market structure, \( v_r \) and \( v_p \) were given by Case (IV) in Lemma A.1. In particular, \( v_r = \frac{R}{\pi_r \alpha(1 - \delta)} \) and \( v_p = v_{nr} + \frac{v_{nr} - p}{\pi_r \alpha v_{nr}} \). The asymptotic expression for \( v_{nr} \) is given in (A.60). Substituting in the vendor’s price, given in (A.61) and simplifying \( r(\sigma^*) \), we have
the equilibrium size of the consumer segment willing to pay ransom in this case is

\[ r_{IV}(\sigma^*) = \frac{1}{2} - \frac{R}{\alpha} + \frac{c_p}{R\pi_r} + \sum_{k=1}^{\infty} a_k \delta^k. \] (A.115)

This is also strictly decreasing in \( \pi_r \) for sufficiently small \( \delta \).

For the total expected ransom paid, this is given as

\[ T_{IV}(\sigma^*) = c_p \left( \frac{1}{2} - \frac{R}{\alpha} + \frac{c_p}{R\pi_r} \right) + \sum_{k=1}^{\infty} a_k \delta^k. \] (A.116)

Under conditions of the focal region and focusing on \( R \) near \( R = \frac{\alpha}{2-c_p} \), we have that \( \frac{d}{d\pi_r} \left[ T_{IV}(\sigma^*) \right] < 0 \).

To complete the proof of Proposition 3, we compare (A.115) to (A.109) at their \( \pi_r \) boundary \( \pi_r = \pi_2 \) to see that, for sufficiently small \( \delta \), \( r_{IV}(\sigma^*) < r_{V}(\sigma^*) \) at the \( \pi_r = \pi_2 \) boundary. Comparing the expected total ransom paid at this boundary, for sufficiently small \( \delta \), we find that \( T_{IV}(\sigma^*) > T_{V}(\sigma^*) \) at that \( \pi_r \) boundary. ■

**Proof of Proposition 4:** From Lemma A.6, under the conditions of this proposition, the consumer market equilibrium outcome is \( 0 < v_{nr} < v_r < v_p < 1 \).

If \( 0 < v_{nr} < v_r < v_p < 1 \) is induced in equilibrium, then from (A.62), the asymptotic expression for the vendor’s profit is given by

\[ \Pi^*_IV = \frac{R - c_p\alpha}{4R} + \frac{c_p\alpha^2(2c_p + R\pi_r)\delta}{8R^3\pi_r} + \sum_{k=2}^{\infty} a_k \delta^k. \] (A.117)

Differentiating with respect to \( \delta \), we have that \( \frac{d}{d\delta} \left( \Pi^*_IV \right) = \frac{c_p\alpha^2(2c_p + R\pi_r)}{8R^3\pi_r} + \sum_{k=1}^{\infty} b_k \delta^k \) for some sequence of coefficients \( b_k \). It follows that for sufficiently small \( \delta \), \( \frac{d}{d\delta} \left( \Pi^*_IV \right) > 0 \).

For the market size to be decreasing in \( \delta \), we will show that the equilibrium \( v_{nr} \) under optimal pricing is increasing in \( \delta \). By substituting in (A.61) into (A.60), the asymptotic expression for this threshold in equilibrium is given by

\[ v^*_{nr} = \frac{1}{2} + \frac{c_p\alpha^2}{8R(R - c_p\alpha)}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \] (A.118)

Taking the derivative of this with respect to \( \delta \) and focusing on the zero-order term, we have that \( \frac{d}{d\delta} [v^*_{nr}] = \frac{c_p\alpha^2}{8R(R - c_p\alpha)} + \sum_{k=2}^{\infty} a_k \delta^k \). This is positive for sufficiently small \( \delta \) since \( R > c_p\alpha \) is a condition of this case (for \( p^*_IV > 0 \)). Hence, the market size shrinks in \( \delta \) for sufficiently small \( \delta \) under the conditions of this case.
The aggregate unpatched loss measure given as

\[ UL \triangleq \int_V \mathbf{1}_{(\sigma^*(v)=(B,NP,NR)}) \pi_r \alpha u(\sigma^*) vd\nu + \int_V \mathbf{1}_{(\sigma^*(v)=(B,NP,R)}) \pi_r u(\sigma^*)(R + \delta \alpha v) dv. \]

Consequently, aggregate unpatched losses are given as

\[ UL_{IV}^* = \int_{v_{nr}}^{v_r} \pi_r \alpha (v_p - v_{nr}) vd\nu + \int_{v_{nr}}^{v_r} \pi_r (v_p - v_{nr})(R + \delta \alpha v) dv. \quad (A.119) \]

Substituting (A.61) into (A.60), we can characterize the equilibrium \( v_{nr} \) threshold as

\[ v_{nr}(p_{IV}^*) = \frac{1}{2} + \frac{c_p \alpha^2 \delta}{8R(R - c_p \alpha)} + \sum_{k=2}^{\infty} a_k \delta^k. \quad (A.120) \]

Then substituting (A.61) and (A.120) into (A.24), we can characterize the equilibrium \( v_p \) threshold as

\[ v_p(p_{IV}^*) = \frac{1}{2} + \frac{c_p}{R \pi_r} + \frac{c_p \alpha (8c_p^2 \alpha + R^2 \pi_r (-4 + \pi_r \alpha) + 4c_p R (-2 + \pi_r \alpha)) \delta}{8R^3 \pi_r (R - c_p \alpha) \pi_r^2} + \sum_{k=2}^{\infty} a_k \delta^k. \quad (A.121) \]

Finally, the asymptotic expansion of \( v_r = \frac{R}{\alpha(1-\delta)} \) is given by

\[ v_r(p_{IV}^*) = \frac{R(1 + \delta)}{\alpha} + \sum_{k=2}^{\infty} a_k \delta^k. \quad (A.122) \]

Substituting in (A.120), (A.122), and (A.121) into the above expression, the asymptotic characterization of the aggregate unpatched losses is given as

\[ UL_{IV}^* = \frac{c_p (8c_p \alpha - (-2R + \alpha)^2 \pi_r)}{8R \pi_r \alpha} + \frac{c_p \left( \frac{R(2R-\alpha)(4R^3 - 4c_p R^2 \alpha + R \alpha^2 - 2c_p \alpha^3)}{\alpha(-R+c_p \alpha)} - \frac{24c_p^2 \alpha}{\pi_r^2} + \frac{2c_p(4R^2 - 8Ra + \alpha^2)}{\pi_r} \right)}{16R^3} \delta + \sum_{k=2}^{\infty} a_k \delta^k. \quad (A.123) \]

Taking the derivative with respect to \( \delta \), we have

\[ \frac{d}{d\delta} [UL_{IV}^*] = \frac{c_p \left( \frac{R(2R-\alpha)(4R^3 - 4c_p R^2 \alpha + R \alpha^2 - 2c_p \alpha^3)}{\alpha(-R+c_p \alpha)} - \frac{24c_p^2 \alpha}{\pi_r^2} + \frac{2c_p(4R^2 - 8Ra + \alpha^2)}{\pi_r} \right)}{16R^3} + \sum_{k=1}^{\infty} b_k \delta^k. \quad (A.124) \]
Under the assumptions of the focal region along with \( \pi_r > \pi \) and \( R > \frac{\alpha}{2-c_p} \), we have that
\[
\frac{c_p \left( \frac{R(2R-\alpha)(4R^3-4c_p R^2 + R\alpha^2 - 2c_p \alpha^3)}{\alpha (R^2 - c_p \alpha^2)} - \frac{2c_p^2 \alpha}{\pi^2} + \frac{2c_p (4R^2 - 8R\alpha + \alpha^2)}{\pi r} \right)}{16R^3} < 0 \text{ for sufficiently small } \delta.
\]

Similarly, denote consumer surplus as \( CS \triangleq \int_{\mathcal{V}} 1_{\{\sigma^*(v) \in \{(B,NP,NR),(B,NP,R),(B,P)\}\}} U(v, \sigma)dv. \)

For the case of \( 0 < \nu_{nr} < v_r < v_p < 1 \), this becomes
\[
CS_{IV}^* = \int_{v_{nr}}^{v_r} (v - p_{IV}^* - \pi_r \alpha (v_p - v_{nr}) v) dv + \int_{v_r}^{v_p} (v - p_{IV}^* - \pi_r \alpha (v_p - v_{nr})(R + \delta \alpha v)) dv + \int_{v_p}^{1} (v - p_{IV}^* - \alpha) dv. \tag{A.125}
\]

Substituting in (A.61), (A.120), (A.122), and (A.121) into the above expression, the asymptotic characterization of consumer surplus is given as
\[
CS_{IV}^* = \frac{1}{8} \left( 1 + c_p \left( -8 + \frac{4R}{\alpha} + \frac{3\alpha}{R} \right) \right) + \frac{c_p (4c_p^2 \alpha^2 + c_p \alpha (-4R^2 + 4R\alpha - 3\alpha^2) \pi_r + R(4R^3 - 2R^2 \alpha + R\alpha^2 - 2\alpha^3) \pi_r^2)}{8R^3 \alpha \pi_r^2} \delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{A.126}
\]

Taking the derivative with respect to \( \delta \), we have
\[
\frac{d}{d \delta} [CS_{IV}^*] = \frac{c_p (4c_p^2 \alpha^2 + c_p \alpha (-4R^2 + 4R\alpha - 3\alpha^2) \pi_r + R(4R^3 - 2R^2 \alpha + R\alpha^2 - 2\alpha^3) \pi_r^2)}{8R^3 \alpha \pi_r^2} + \sum_{k=2}^{\infty} b_k \delta^k. \tag{A.127}
\]

We will show that \( \frac{c_p (4c_p^2 \alpha^2 + c_p \alpha (-4R^2 + 4R\alpha - 3\alpha^2) \pi_r + R(4R^3 - 2R^2 \alpha + R\alpha^2 - 2\alpha^3) \pi_r^2)}{8R^3 \alpha \pi_r^2} < 0 \) under the conditions of the proposition. This is equivalent to \( 4c_p^2 \alpha^2 + c_p \alpha (-4R^2 + 4R\alpha - 3\alpha^2) \pi_r + R(4R^3 - 2R^2 \alpha + R\alpha^2 - 2\alpha^3) \pi_r^2 < 0 \). This is a quadratic in \( \pi_r \), with negative second-order term for \( R \) sufficiently close to the \( R = \frac{\alpha}{2 - c_p} \) boundary (\( R_2 \)). For this to hold, either there are two real roots and \( \pi_r \) is larger than the larger root of that quadratic or smaller than the smaller root of that quadratic, or there are not two real roots (in which case the inequality is always satisfied). Under the conditions the proposition, the larger of the two roots is given by
\[
\pi_{r,2} = \frac{8c_p \alpha}{4R^2 - 4R\alpha + 3\alpha^2 + \sqrt{-48R^4 + 24R^2 \alpha^2 + 8R\alpha^3 + 9\alpha^4}}. \tag{A.128}
\]

A.44
Under the conditions of the proposition, the smaller root is negative, so we want to show that \( \pi_r > \pi_{r,2} \) in this case. Note that \( \frac{c_p \alpha}{R^2 - c_p R \alpha^2} > \pi_{r,2} \) from the conditions of the proposition and the focal region assumptions, and since \( \pi_r > \frac{c_p \alpha}{R^2 - c_p R \alpha^2} \) for this proposition, it follows that \( \pi_r > \pi_{r,2} \). Therefore, \( \frac{d}{d\delta} [CS_{IV}^*] < 0 \) for sufficiently small \( \delta \) under the conditions of the proposition. \[\Box\]

**Proof of Proposition 5:** From the consumer utility function (A.1), a consumer of valuation \( v \) prefers (B, NP, R) over (B, NP, NR) if and only if \( v - p - \pi_r u(\sigma)(R + \delta \alpha v) \geq v - p - \pi_r u(\sigma)v \). This is equivalent to \( v \geq \frac{R}{\alpha(1-\delta)} \). Consequently, if \( \frac{R}{\alpha(1-\delta)} > 1 \) (or \( \delta > 1 - \frac{R}{\alpha} \)), then no consumer would prefer (B, NP, R) over (B, NP, NR).

As (B, NP, R) is a strictly dominated option under this condition, consumers are left with (NB), (B, NP, NR), and (B, P) as incentive-compatible choices. Consequently, when \( \delta > 1 - \frac{R}{\alpha} \), the consumer market equilibrium characterization no longer depends on \( R \) or \( \delta \), and the characterization of the consumer market equilibrium is given in Lemma A.3.

In particular, if Case (I) of Lemma A.3 holds, then \( 0 < v_{nr} < 1 \) would be the equilibrium outcome. From the first-order condition, the interior optimal price of this case is given by \( p_1^* = \frac{-1 + \pi_r \alpha(1-\pi_r \alpha) + \sqrt{1 + \pi_r \alpha + (\pi_r \alpha)^2 + (\pi_r \alpha)^4}}{9 \pi_r \alpha} \). If \( \pi_r \alpha > \frac{(2 - 3c_p) c_p}{1 - 2c_p} \), then \( p_1^* \) will not satisfy the conditions in (I). Note that \( \frac{(2 - 3c_p) c_p}{1 - 2c_p} > 0 \) holds under the conditions of the focal region. Hence, if \( \pi_r \alpha > \frac{c_p (2 + 3c_p)}{1 + 2c_p} \) and \( \delta > 1 - \frac{R}{\alpha} \), then \( 0 < v_{nr} < v_p < 1 \) is the equilibrium outcome, and no measures of interest change in \( R \) or \( \delta \).

On the other hand, if one of the conditions of Case (II) of Lemma A.3 is: \( c_p + (-1 + c_p + p) \pi_r \alpha < c_p^2 \). Therefore, if \( \alpha < \frac{c_p}{\pi_r \alpha} \), then this condition cannot be met by any price \( p \), including \( p = 0 \). Consequently, if \( \alpha < \frac{c_p}{\pi_r} \) and \( \delta > 1 - \frac{R}{\alpha} \), then the equilibrium outcome will be \( 0 < v_{nr} < 1 \), and no measures of interest change in \( R \) or \( \delta \). \[\Box\]

**Proof of Proposition 6:** Lemma A.9 provides the characterization of the consumer market equilibrium under optimal pricing as \( R \) changes. To complete the proof, we compute the vendor’s price, profit, and market size for each market outcome to do comparative statics with respect to \( R \).

When \( 0 < v_r < 1 \) is induced in equilibrium, the vendor’s price is given in (A.93). The asymptotic expansion in \( \delta \) for the vendor’s profit at this price is given in (A.94). The market size is \( M_V = 1 - v_r \), and the equilibrium \( v_r \) can be found by substituting (A.93) into (A.92). This has an asymptotic expansion given by

\[
M_V^* = \frac{1}{2(1 + R \pi_r)} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{A.129}
\]

Taking the derivatives with respect of each of these with respect to \( R \), we have that the vendor’s equilibrium price increases in \( R \) while the profit and market size shrinks in \( R \).

For the cases of \( 0 < v_{nr} < v_r < 1 \) and \( 0 < v_{nr} < 1 \), the equilibrium prices are given in (A.90)
and (A.88), respectively. Note that they are equal, and neither depend on $R$. Hence, as $R$ changes, the equilibrium price remains constant across $R$ in these regimes. It follows that the vendor’s equilibrium profit is constant in $R$ as well. Also, since $v_{nr}$ does not directly depend on $R$ in either market outcome (and since it does not indirectly depend on $R$ through the price $p$), it follows that the market size is also constant in $R$. ■

**Proof of Proposition 7:** Lemmas A.9 and A.10 provides the characterization of the consumer market equilibrium under optimal pricing as $\pi_r$ changes. Lemma A.9 states that when $R \leq \tilde{R}_2$, the equilibrium consumer market structure is $0 < v_r < 1$. Lemma A.10 states that for a higher range of $R$ (specifically, $\tilde{R}_2 < R < \tilde{R}_3$), the market structure is $0 < v_{nr} < v_r < 1$ for $\pi_r < \tilde{\pi}$ and is $0 < v_r < 1$ otherwise. To complete the proof, it suffices to do comparative statics on the equilibrium price for the three cases. Then we will compare the prices.

For the case of $0 < v_r < 1$, the price comparative statics are the same as provided in Proposition 2. In particular, for sufficiently low $\delta$, the price decreases in $\pi_r$ if $\pi_r < \frac{1}{R}$. This implies that the price decreases for all $\pi_r \in (0, 1)$ if and only if $R < 1$. Since $R$ is bounded above by $\tilde{R}_2$, for sufficiently low $\delta$, $R > 1$ can only happen is $\alpha > 2$. Hence, if $\alpha > 2$ and $R > 1$, then the vendor’s price decreases in $\pi_r$ for $(0, \frac{1}{R})$ and increases in $\pi_r$ for $\pi_r > \frac{1}{R}$. Otherwise, if either $\alpha \leq 2$ or $R \leq 1$, then the vendor’s price will decrease in $\pi_r$ for all $\pi_r$.

For the case of $0 < v_{nr} < v_r < 1$, the equilibrium price is given in (A.88). The price is decreasing in $\pi_r$ as long as $\pi_r < \frac{1}{\alpha}$. We want to show that $\frac{1}{\alpha} < \pi'$. For sufficiently low $\delta$, this is equivalent to $R < \frac{11\alpha}{16}$. For sufficiently small $\delta$ and $R$ sufficiently close to $R = \tilde{R}_2$, so there exists an $\tilde{R}_3$ such that for $R \in (\tilde{R}_2, \tilde{R}_3)$, we have that $R < \frac{11\alpha}{16}$. Hence, the price is decreasing in $\pi_r$ for $R$ close to $R = \tilde{R}_2$.

Repeating the same argument as in Proposition 2 by comparing the prices (which are the same expressions as in Proposition 2), we have that there is a price hike at $\pi_r = \pi'$.

**Proof of Proposition 8:** Lemma A.10 provides the characterization of the consumer market equilibrium under optimal pricing as $\pi_r$ changes for $R \in (\tilde{R}_2, \tilde{R}_3)$, and Lemma A.9 provides the consumer market outcome characterization for $R \leq \tilde{R}_2$. To complete the proof, we compute the vendor’s equilibrium profit, market size, the size of the population willing to pay ransom, and the expected total ransom paid.

For the case of $0 < v_r < 1$, the expected total ransom paid is increasing in $\pi_r$ for sufficiently low $\delta$ if $R\pi_r < 1$. This is the same condition as for the price comparative statics for the same case in the previous proposition. In particular, for sufficiently low $\delta$, expected total ransom paid increases in $\pi_r$ for all $\pi_r$ if $\alpha \leq 2$ or if $\alpha > 2$ and $R \leq 1$. Otherwise, the expected total ransom paid will be non-monotonic in $\pi_r$.

For $0 < v_{nr} < v_r < 1$, the the size of the ransom-paying population is $r(\sigma) = 1 - v_r$. Since $v_r = \frac{R}{\alpha(1-\delta)}$ in this case is constant in $\pi_r$, it follows that the size of the ransom-paying population is constant in $\pi_r$ in this case.

The vendor’s profit is given in (A.91). The equilibrium market size is $M = 1 - v_{nr}$, where the equilibrium $v_{nr}$ can be found by substituting (A.90) into (A.89). Taking the derivative
of these with respect to \( \pi_r \), we see that both the market size and vendor's profit decreases in \( \pi_r \) for any \( \alpha > 0 \) and \( \pi_r > 0 \).

The expected total ransom paid in this case is \( T(\sigma) = \pi_r(1 - v_{nr})(1 - v_r)R \). Using \( v_r = \frac{R}{\pi(1 - \delta)} \) and substituting in the optimal price (A.90) into the expression for \( v_{nr} \) in this case (A.89), we have that the expected total ransom paid has an asymptotic expansion in \( \delta \):

\[
T_{III}(\sigma^*) = R(R - \alpha) \left( -3 - 3\alpha \pi_r + \sqrt{5 + \pi_r \alpha(-2 + 5\pi_r \alpha)} + 4\sqrt{1 + \pi_r \alpha + (\pi_r \alpha)^3 + (\pi_r \alpha)^4} \right) + \frac{6\alpha^2}{\pi_r} \sum_{k=1}^{\infty} a_k \delta^k. \quad (A.130)
\]

Using \( R < \alpha \) (which follows from \( R < \tilde{R}_3 \)) and \( R > \frac{1}{2} \alpha \), we have that \( \frac{d}{d\pi_r} [T(\sigma^*)] > 0 \).

For \( 0 < v_r < 1 \) when \( R > \tilde{R}_2 \), the vendor's equilibrium profit is given in (A.94). The equilibrium market size is \( M = 1 - v_r \), where the equilibrium \( v_r \) can be found by substituting the price (A.93) into the expression of \( v_r \) in this case (A.92). The asymptotic expression in \( \delta \) is given as

\[
M_V(\sigma^*) = \frac{1}{2(1 + R\pi_r)} + \sum_{k=1}^{\infty} a_k \delta^k. \quad (A.131)
\]

Taking the derivative of the market size and profit with respect to \( \pi_r \), we see that these measures decrease in \( \pi_r \) in this case.

Also, since \( r(\sigma) = 1 - v_r \) is the same as the market size in this case, it follows that the population willing to pay ransom decreases in \( \pi_r \). The expected total ransom paid is \( T(\sigma) = \pi_r n(\sigma) r(\sigma) R = \pi_r(1 - v_r)(1 - v_r) R \). The asymptotic expression in \( \delta \) for this is given below.

\[
T_V(\sigma^*) = \frac{R\pi_r}{4(1 + R\pi_r)^2} + \sum_{k=1}^{\infty} a_k \delta^k. \quad (A.132)
\]

That this is increasing in \( \pi_r \) for sufficiently small \( \delta \) holds for \( \pi_r < \frac{1}{R} \). If \( R \leq 1 \), then the expected total ransom paid is increasing in \( \pi_r \) for all \( \pi_r \). Otherwise, if \( R > 1 \), then the expected total ransom paid increases in \( \pi_r \) for \( \pi_r < \frac{1}{R} \) and decreases in \( \pi_r \) for \( \pi_r > \frac{1}{R} \).

Finally, since there is a price hike at \( \pi_r = \pi' \) from Proposition 7, the market size shrinks at that discontinuity so the usage risk shrinks due to the price hike. Consequently, the expected total ransom paid decreases due to the price hike. This completes the proof. ■

**Proof of Proposition 9:** To compare the benchmark and ransomware scenarios, we will use a common \( \pi \) parameter to denote the risk factor. In the ransomware case, \( \pi = \pi_r \), and in the benchmark case, \( \pi = \pi_n \).

A.47
We will show that for sufficiently low $\pi$, the equilibrium price of the benchmark case and the equilibrium price of the ransomware case match. Then we will show that for an intermediate range of $\pi$, the equilibrium price of the ransomware case is greater than the price of the benchmark scenario. Lastly, we will show that for a high range of $\pi$, the equilibrium price of the benchmark case is greater than that of the ransomware case.

First, by Proposition 5, for sufficiently low $\pi$ in the benchmark case, we have $0 < v_n < 1$ as the equilibrium outcome (in which nobody patches). By Lemma A.10, for sufficiently low $\pi$, we have that $0 < v_{nr} < v_r < 1$ is the equilibrium outcome under the conditions of the proposition. Comparing the prices (A.55) and (A.57), we note that they are equal, so for sufficiently low $\pi$, the equilibrium price of the benchmark case and the equilibrium price of the ransomware case match.

Next, we will show that under the conditions of the proposition, whether $0 < v_n < 1$ or $0 < v_n < v_p < 1$ arises under the benchmark case, we will have $p^*_{RW} > p^*_{BM}$ when $0 < v_r < 1$ is the induced equilibrium outcome. If $0 < v_n < 1$ is the equilibrium outcome under the benchmark case, then the equilibrium price would be the same as that of $0 < v_{nr} < v_r < 1$ (again, comparing the prices (A.55) and (A.57)). We showed in Proposition 2 that there is a price jump at the point of discontinuity. Hence, if $0 < v_n < 1$ is still the equilibrium outcome under the benchmark case, then $p^*_{RW} > p^*_{BM}$.

On the other hand, suppose that $0 < v_n < v_p < 1$ is the equilibrium outcome under the benchmark case. At the value of $\pi$ at which point the vendor switches price to induce $0 < v_n < v_p < 1$ instead of $0 < v_n < 1$ in the benchmark case, it cannot be the case that the vendor does so by hiking the price. This is because a price hike would restrict unpatched usage, making patching even less incentive-compatible than with lower $\pi$. Therefore, a regime switch must be accompanied by a price drop, and the vendor’s price in $0 < v_n < v_p < 1$ is lower than the price in the case of $0 < v_n < 1$ (which was shown to be lower than the price in the ransomware case). To sum up, whether $0 < v_n < 1$ or $0 < v_n < v_p < 1$ arises under the benchmark case, we will have $p^*_{RW} > p^*_{BM}$ when $0 < v_r < 1$ is the induced equilibrium outcome.

Lastly, consider when $\pi > \pi_2$ of Lemma A.7. By Lemma A.7, the equilibrium outcome under the benchmark case is $0 < v_{nr} < v_r < 1$. By Proposition 5, we have that if $\pi > \frac{c_\alpha(2-3c_\alpha)}{c_\alpha(1-2c_\alpha)}$, then the equilibrium outcome under the benchmark case if $0 < v_n < v_p < 1$. Under the conditions of the focal region, when $R$ is close to the boundary $R = R_2$, then $\pi_2 > \frac{c_\alpha(2-3c_\alpha)}{c_\alpha(1-2c_\alpha)}$. Hence, when $R$ is close to $R = R_2$ and $\pi > \pi_2$, then the equilibrium outcome under the ransomware case is $0 < v_{nr} < v_r < v_p < 1$ while the equilibrium outcome under the benchmark case is $0 < v_n < v_p < 1$.

We compare the price of $0 < v_{nr} < v_r < v_p < 1$ given in (A.61) to the benchmark price given in (A.47). Since $v_{nr} > \frac{1}{2}$ by Lemma A.4 and $p^*_{II}(v_{nr})$ is increasing in $v_{nr}$ (shown in the proof of that lemma), it follows that a lower bound on $p^*_{II}$ is $p^*_{II} \left( \frac{1}{2} \right) = \frac{1}{8} \left( 4 + \pi_r \alpha - \sqrt{\pi_r \alpha (16c_p + \pi_r \alpha)} \right)$. The expression $\frac{1}{8} \left( 4 + \pi_r \alpha - \sqrt{\pi_r \alpha (16c_p + \pi_r \alpha)} \right) > \frac{R - c_\alpha}{2R}$ (where the right side of the inequality comes from the constant term of the asymptotic ex-
pansion for the price in $0 < v_{nr} < v_r < v_p < 1$ is equivalent to $2c_p\alpha + R(-2R + \alpha)\pi_r > 0$. This holds for all $\pi_r$ for $R$ sufficiently close to $R = R_2$ under the conditions of the focal region. Hence, the benchmark price is greater than the ransomware price for $\pi > \pi_2$ of Lemma A.7.

**Proof of Proposition 10:** Similar to Proposition 9, we will use the $\pi$ notation to denote a risk factor parameter across both the ransomware scenario as well as the benchmark scenario, and the proof of this proposition follows a similar structure.

Under the ransomware scenario, the consumer market equilibrium across $\pi$ was given in Lemma A.7. When $\pi$ is low in the ransomware scenario, then the equilibrium outcome is $0 < v_{nr} < v_r < 1$ under the conditions of the proposition. By Proposition 5, the consumer market outcome under the benchmark scenario is $0 < v_n < 1$.

When the equilibrium market outcome is $0 < v_{nr} < v_r < 1$, for sufficiently small $\pi_r$, the equilibrium welfare is given as

$$SW_{III} = \frac{3}{8} + \frac{(R^2 - 2Ra)\pi_r}{4\alpha} + \sum_{k=1}^{\infty} a_k\pi_r^k. \quad (A.133)$$

For $0 < v_n < 1$, the asymptotic expression for the welfare in $\pi_r$ for sufficiently small $\pi_r$ is given by

$$SW_I = \frac{3}{8} - \pi_r\alpha \frac{\alpha}{4} + \sum_{k=1}^{\infty} a_k\pi_r^k. \quad (A.134)$$

Comparing against (A.134), we have that ransomware dominates the benchmark for sufficiently low $\pi_r$. Therefore, there exists a bound an $\pi_L > 0$ such that if $0 \leq \pi < \pi_L$, then $SW_{RW} \geq SW_{BM}$ for sufficiently low $\delta$.

Next, consider the high $\pi$ case. Under ransomware, the equilibrium outcome for $\pi > \pi_2$ (from Lemma A.7) is $0 < v_{nr} < v_r < v_p < 1$. As shown in the proof of Proposition 9, under this high $\pi$ range, the equilibrium outcome under the benchmark would be $0 < v_n < v_p < 1$. Under the ransomware scenario, the asymptotic expression in $\delta$ of the equilibrium welfare of this case is given as

$$SW_{IV} = \frac{1}{8} \left( 3 + c_p \left( -8 + \frac{4R}{\alpha} + \frac{\alpha}{R} \right) \right) + \sum_{k=1}^{\infty} a_k\delta^k. \quad (A.135)$$

On the other hand, in the benchmark case, the equilibrium price satisfies (A.47).

We also have that $v_p = \frac{c_p v_{nr}}{v_{nr} - p}$ from (A.14). The welfare of this case is given by

$$SW_{II} = \int_{v_{nr}}^{v_p} (v - \pi_r\alpha(v_p - v_{nr})v) dv + \int_{v_p}^{1} (v - c_p) dv. \quad (A.136)$$

Substituting in (A.14) for $v_p$ and substituting in (A.47) for the price as a function of $v_{nr},$
we have that the welfare expression as a function of $v_{nr}$ is given as

$$SW_{II}(v_{nr}) = \frac{1}{4} \left( v_{nr}^2 \left( \sqrt{\alpha \pi_r} \sqrt{4c_p + \alpha \pi_r v_{nr}^2} + \alpha \pi_r (-v_{nr} - 2) \right) + c_p \left( \frac{\sqrt{4c_p + \alpha \pi_r v_{nr}^2}}{\sqrt{\alpha \pi_r}} + v_{nr} - 4 \right) + 2 \right). \quad (A.137)$$

Using $v_{nr} > \frac{1+c_p^2}{2}$ from Lemma A.4, that (A.137) is strictly decreasing in $v_{nr}$ for $v_{nr} > \frac{1+c_p^2}{2}$ follows from the conditions on $c_p$ and $\alpha$ of the focal region. Therefore, a lower bound on social welfare of this benchmark case is (A.137) evaluated at $v_{nr} = \frac{1+c_p^2}{2}$, and an upper bound is (A.137) evaluated at $v_{nr} = \frac{1+c_p^2}{2}$.

Focusing on $R$ sufficiently close to $R = R_2$, comparing the first-order term of (A.135) to (A.137) evaluated at $v_{nr} = \frac{1+c_p^2}{2}$, we have that the welfare of the ransomware case is greater than the upper bound on social welfare if $\alpha > 2c_p + 1$ along with the conditions in the focal region. This is a non-empty region of the parameter space since $2c_p + 1 < 2(2 - c_p)^2$ for all $0 < c_p < 2 - \sqrt{3}$. Hence, if $\alpha > 2c_p + 1$, the welfare of the ransomware case is higher than the welfare of the benchmark case for $\pi$ close to 1.

Lastly, consider an intermediate range of $\pi$, for $\pi$ in a range $(\pi_2 - \epsilon, \pi_2)$ (where $\pi_2$ comes from Lemma A.7). By Lemma A.7, the equilibrium market outcome under the ransomware case is $0 < v_r < 1$. Then the equilibrium $v_r$ has an asymptotic expression given by

$$v_r^* = \left( 1 - \frac{1}{2 + 2R\pi_r} \right) + \sum_{k=1}^{\infty} a_k \delta^k. \quad (A.138)$$

The welfare function of this case is given by

$$SW \triangleq \int_V I_{\{\sigma^*(v) = (B,NP,R)\}} \left( v - \pi_r u(\sigma^*)(R + \delta\alpha v) \right) dv.$$  

Consequently, welfare is given as

$$SW_{v}^* = \int_{v_r}^{1} (v - \pi_r (1 - v_{nr})(R + \delta\alpha v)) dv. \quad (A.139)$$

Substituting in (A.138) into the above expression, the asymptotic characterization of the welfare is given as

$$SW_{v}^* = \frac{3 + 2R\pi_r}{8(1 + R\pi_r)^2} + \sum_{k=1}^{\infty} a_k \delta^k. \quad (A.140)$$
In particular, when $\pi_r = \pi_2$, then

$$SW^*_V = \frac{(R - c_p\alpha)(3R - c_p\alpha)}{8R^2} + \sum_{k=1}^{\infty} a_k\delta^k.$$  \hspace{1cm} (A.141)

On the other hand, in the benchmark case, the welfare of $0 < v_{nr} < v_p < 1$ will be bounded below by (A.137) evaluated at $v_{nr} = \frac{1+c_p}{2}$. Comparing (A.141) with $SW^*_V|_{v_{nr} = \frac{1+c_p}{2}}$ at $R$ sufficiently close to $R = R_2$, there exists some $\epsilon > 0$ such that the welfare under the ransomware case is less than that of the benchmark case for $\pi_r \in (\pi_2 - \epsilon, \pi_2)$. Together with the other two cases, this proves the proposition statement. ■