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Yifan Dou *
School of Management
Fudan University
Shanghai 200433, China
yfdou@fudan.edu.cn.

Hao Hu
Scheller College of Business
Georgia Institute of Technology
Atlanta, GA 30308
haohu@gatech.edu.

Marius F. Niculescu *
Scheller College of Business
Georgia Institute of Technology
Atlanta, GA 30308
marius.niculescu@scheller.gatech.edu.

D. J. Wu *
Scheller College of Business
Georgia Institute of Technology
Atlanta, GA 30308
dj.wu@scheller.gatech.edu.

* Corresponding Authors
All authors had equal contribution. Author names listed in alphabetical order.

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Abstract

In software markets, the sheer number of available applications makes it rather challenging for any given new one to stand out and be noticed by consumers. Moreover, a push towards privacy by regulators and consumers is making it harder to target consumers. As such, firms have to rely on more non-targeted go-to-market strategies. We explore two popular strategies through which developers can catalyze adoption by helping consumers directly or indirectly learn the value of their products - *seeding* (free full-feature product giveaways to a subset of the consumer base) and *time-limited freemium* (*TLF*). Seeding, as a business strategy, existed for a long time. On the other hand, the feasibility to offer market-wide *TLF* became mainstream more recently, with the advent of the Internet and a plethora of digital tools. Thus, a natural question emerges - if *TLF* represents nowadays a feasible and easily implementable strategy for software applications, has seeding approach been rendered irrelevant in these markets? In this study, we provide managerial recommendations on when each of these strategies with a free full-feature-consumption component is optimal, based on social and self-learning dynamics, consumer priors, adoption costs, and individual product value depreciation. To that end, under a multi-period parsimonious unifying framework, we show that S becomes dominated as free trials enter the picture. We identify two specific market factors that, when present, can induce seeding to be optimal when consumers initially underestimate true product value - (i) user adoption costs and/or (ii) individual depreciation of value by usage. Moreover, we show that these two factors have a moderating effect on the impact of word-of-mouth (WOM) effects on the optimality of seeding. In the absence of these factors, stronger WOM effects alone cannot give seeding an edge against the other business strategies. However, once either depreciation or adoption costs are accounted for, strong WOM effects increase the relevance of seeding (enlarging its optimality region in the parameter space). Our results remain qualitatively consistent under a battery of robustness checks.

Keywords: seeding, time-limited freemium, adoption costs, word-of-mouth effects, individual depreciation, social learning.

1 Introduction

The software app markets have experienced tremendous growth during the last decade thanks to the advances in Internet technologies, the widespread use of desktop and mobile devices, and a lower entry barrier for developers. Microsoft, for Windows 10 alone, has facilitated compatibility with "over 35 million application titles with greater than 175 million application versions, and 16 million unique hardware/driver combinations" (Fortin 2018). In the mobile space, as of Apr. 2025, the top two app stores, Google Play and Apple App Store, boasted a combined app count above 3.4 million (Roth 2025).

However, in today's saturated app market, significant profits (and thus market success) can be elusive for developers. Recent analyses report that fewer than 20% of mobile apps earn \$1K per month within their first two years, while the top 1% of publishers capture over 90% of global mobile app revenue (Cruz 2022, RevenueCat 2025). A major challenge for both desktop and mobile app startups is gaining traction early, in the critical stages of the adoption process (Gokgoz et al. 2021). It is well established that consumers adopt software based on their own initial perceptions of the product's value - often referred to as "priors" - which may align with or deviate substantially from the actual real product value (Weathers et al. 2007, Shulman et al. 2015, Chen et al. 2021, Zhang et al. 2022). Consumers can update their valuation of the product via several learning mechanisms. On one hand, consumers can engage in *social learning* via word-of-mouth (WOM), allowing their perceptions to be shaped to a certain degree by the opinions of other consumers or experts. On the other hand, if consumers interact with the product directly, they can engage in *self-learning*, whereby they update their priors on the value of the product upon using it for a period of time. The emergence of widely available generative artificial intelligence (GenAI) assistants and agents is showing great potential to alter this valuation learning process even further.

Understanding the potential pre-adoption misalignment between consumer valuation perceptions and real valuations, as well as the dynamics of the consumer valuation discovery process, software producers have increasingly embraced various forms of *free*-consumption to steer consumer learning and induce revenue-generating adoption. Such strategies are particularly salient in the context of *experience goods* - a broad category which encompasses many digital goods whose value and fit are better understood by consumers once they are directly exposed to the product/service. Two popular strategies employing the free-consumption approach are *seeding* (S) and *time-limited freemium* (TLF, otherwise referred to as time-locked *free trials*).

Through S, developers provide the full-functionality product for free to a subset of the market, counting on these seeded consumers to not only use the product but also help spread awareness and knowledge about it within their respective communities and beyond. Seeding as a business strategy has existed for a long time, since before the emergence of digital goods. What software seeding adds to the traditional seeding model is the potential for scale (and, hence, more fine-tuning) given negligible marginal costs and the ability to reach via the Internet the entire addressable market. There are many instances of software products being offered for free for non-commercial use and for a fee for commercial use. For example, many providers such as IBM, Microsoft, and SAS offer a bundle of their developer-grade products for free to students and educators. Via its Technology Impact Program, Autodesk donates free licenses for many of its products to nonprofits, startups, and entrepreneurs that use design for environmental or social good. Seeding is also a popular strategy within mobile app markets via free app giveaways (pushed through portals such as AppAdvice, AppsFree, and Giveaway of the Day). Moreover, seeding and price discounts for a variety of products have been used as popular, albeit frowned-upon incentives by market entrants without established brands to harvest online reviews to jumpstart WOM effects (Hautala 2022).

Under *TLF*, all consumers are able to try the full-functionality product at no charge during a limited trial period, after which they are required to pay for continued use. TLF is a relatively recent business strategy - while TLF draws its roots from traditional product sampling, the feasibility to implement a consistent, market-wide TLF strategy has truly been ushered in by the advent of digital goods and services for it relies on encapsulating a limited free-for-all consumption component with *digitally encoded automatic expiration* at the end of the trial period. Free trial windows typically span from a few days to a few months. TLF strategies have been employed for many categories of apps and services in domains including engineering and design (e.g., AutoCAD, VeSys, Adobe Creative Cloud), productivity (e.g., Salesforce CRM products, Microsoft Office 365), IT security (e.g., Crowdstrike, Norton, Bitdefender), content provision (e.g., Hulu, Apple TV+, Tidal, Audible), health and wellness (e.g., Peloton, Calm, Nutrium), professional and personal education (LinkedIn Learning, Rosetta Stone, Pluralsight), just to name a few. In the mobile app market, Google Play Store and Apple App Store both allow for native implementation of TLF for subscription-based apps (since 2012 and 2017, respectively). With the largest mobile app marketplaces nowadays aligned in supporting TLF, accounting for free trials within the go-to-market strategy choice set for mobile app developers is of practical and timely managerial relevance. For a broader discussion of how our analysis is relevant to the mobile sector, please see E-companion I.

Recent privacy regulations across major jurisdictions (e.g., the EU's General Data Protection Regulation; the California Consumer Privacy Act and Colorado Privacy Act in the U.S.), combined with platform-level initiatives (e.g., Apple's App Tracking Transparency) and rising consumer awareness and proactivity toward privacy (Cisco 2024), have sharply curtailed data-driven marketing (Aridor et al. 2025), prompting firms to revisit less targeted, more naïve approaches. If firms employ seeding, a commonly held opinion is that they would prefer to target specific customers if they could - but what if they cannot do that? In this study, we zero in on *non-targeted* seeding, and ask the following overarching research question: *Does non-targeted seeding still merit inclusion in a software firm's go-to-market strategy portfolio, given that firms can resort to free trials and paid (perpetual or subscription-based) licenses?* Recent studies suggest that random seeding, with slightly larger seed sets, can rival targeted seeding (Akbarpour et al. 2023). While extant literature explored how seeding and free trials could be individually optimized and how they fared against other strategies, surprisingly little research exists on how these two go-to-market approaches fare *against each other* as mechanisms to jumpstart paid adoption. This work addresses this gap. Building on the unifying modeling framework from Niculescu and Wu (2014), we first show that in a parsimonious model incorporating social and self-learning but abstracting away from other adoption and usage factors, S always comes short relative to other considered strategies. While, absent TLF, S has been previously shown to be optimal in regions where consumers initially significantly underestimate the value of the app (Niculescu and Wu 2014), this is no longer the case when free trials enter the picture. This is due to how S and TLF differ in their leveraging of demand cannibalization as well as social vs self-learning, as detailed in Section 3.3. The inability, under S, to monetize seeded customers later on is part of what gives TLF a decisive edge in the baseline setup.

But should we take as a foregone conclusion that non-targeted S is always a dominated strategy and just ignore it? As a second research objective of our work, we seek to *identify specific factors that, when accounted for, support market scenarios under which S is the optimal strategy, even in the presence of TLF*. By identifying such factors, we aim to add further nuanced richness to the theories around go-to-market strategies when consumers learn their own product valuations. Offering some form of free access to the full product involves a delicate balance act, due among others to (i) intrinsic *adoption costs* and (ii) the potential for *value depreciation with use*. Hence, we found it a natural starting point to focus on the impact that these two factors have on the optimality of various developer strategies. Adoption costs[1] can undermine the effectiveness of both TLF and S (and of other strategies as well) as some consumers may shy away from exploring the product in the first place even when there is some free access to it. That being said, when customers initially significantly undervalue the product, S has the edge over TLF due to seeded customers receiving a perpetual license which extends beyond the duration of a free trial, yielding higher willingness to take advantage of the free offer. Value depreciation through use, otherwise referred to as *individual depreciation* (Dou et al. 2017),[2] can also dilute the benefit of free trials (TLF) should consumers be able to utilize the software for a significant portion of their needs before the trial ends. Under S, when customers significantly underestimate the value of the product, the developer gets better opportunities (relative to the other models) to monetize future periods by delaying most paid adoption beyond initial period, allowing seeds to spread WOM and the rest of customers to update their priors upwards, ensuring that a substantial portion of the unseeded consumer population has not yet depleted value through usage before reaching higher willingness

[1]Commonly associated with installation, integration, setup, testing, learning curve, etc. Furthermore, as software applications are getting more complex in terms of features and functionality, the size of their installation footprint has increased considerably, which is particularly challenging for mobile users whose phones have limited storage. If mobile users do not have enough space to install a new app on their device, in order to explore a free-trial app they must either upgrade their cloud backup storage or delete other applications, both of which are costly actions. Also, newer applications may be more resource-demanding, requiring hardware upgrades.

[2]Not to be confused with *obsolescence*, which captures time-based depreciation in value, regardless of usage.

to pay (WTP) through social learning. Individual depreciation is present when the user's need is limited in scope and scale in general.[3] Furthermore, in the mobile space, users tend to lose interest in many installed apps relatively quickly and the retention rates drop to single-digit percentages for the majority of app categories after only one month (Statista 2025). Individual depreciation is also present when consumption is more hedonic (e.g., video games, music, movies), switching costs are negligible, and consumers constantly search for the "next" great experience.[4] On the other hand, tactical enterprise applications that are used for daily operations (e.g., ERP systems, EMR systems, payment systems, cloud storage, IT security solutions) are likely to exhibit low individual depreciation as their value to consumers is not expected to decline through use.

We do confirm that these two factors (considered separately and together) lead to outcomes in which non-targeted seeding can dominate TLF (and other non-free models) when customers significantly underestimate the value of the product. While our study does not completely dispute the continued viability of more naïve seeding as a strategy in today's markets for digital goods, we find that more stars need to align to warrant its use. Our results remain consistent under multiple robustness checks (including presence of both depreciation and adoption costs, endogenous depreciation, heterogeneous priors, generalized WOM effects, imperfect learning, and compounded learning over multiple periods). Interestingly, we show that the two aforementioned additional market factors are also moderators for the effect of WOM on the optimality of the seeding strategy. In their absence, WOM effects alone do not help the seeding strategy dominate the other strategies. Nevertheless, if adoption costs or individual depreciation are present, stronger WOM effects do lead to a larger region of the parameter space where S dominates. In addition to advancing the theory around market seeding strategies (and, more generally, around strategies with a free component), our insights carry significant managerial relevance: they inform when non-targeted seeding should still be considered, and more generally when several go-to-market strategies are optimal, contingent on parameter space regions. Lastly, we show that the advent of GenAI can shift the market into narrower regions of the parameter space in which free-consumption strategies including seeding can particularly shine, further underscoring the timely relevance of our study.

[3] E.g., installing audio editing software for a small project to remove background noise from a handful of tracks, or installing photo editing software to remove dust spots from digital photos from a vacation trip, when the consumer realizes ex-post that the camera sensor was not clean of debris when pictures were taken.

[4] For example, in the context of video games, it has been documented that, on average, players tire quickly of a particular game. According to Shiller (2013), consumers reduce their valuation from \$80 in the first month of use to just a couple of dollars after six months. In fact, after only the first week of ownership, the consumption value that owners place on the games they own already deteriorates between 22% and 49% (Ishihara and Ching 2019).

2 Literature

Our novel theoretical contributions lie predominantly within the space of the *economics of free*, advancing the research agenda on the impact and optimality of *seeding* strategies. For brevity, the discussion in this section centers on this core literature. For completeness, we present in E-companion H the related literature on free trials - a directly connected but secondary stream relative to our main research focus. At the same time, we do acknowledge that our modeling framework integrates modeling elements from several complementary literatures (including multi-period adoption of digital goods, impact of WOM effects on adoption, consumer valuation discovery, and individual use-based value depreciation). Relevant works in these ancillary research streams are referenced throughout the main body of the paper, as we introduce various go-to-market models.

The literature on the seeding business model is rich, exploring various related research questions including optimality of such strategies. At a market level, abstracting from the network structure, several studies employed adaptations of the Bass (1969) model to explain how firms can employ seeding to jumpstart and accelerate the product diffusion process (Jain et al. 1995, Lehmann and Esteban-Bravo 2006, Jiang and Sarkar 2010). Another segment of this literature focuses on how to optimize (or nearly optimize) targeted or stochastic seeding strategies contingent on the topology of the network and the optimization objective (Galeotti and Goyal 2009, Haenlein and Libai 2013, Libai et al. 2013, Schlereth et al. 2013, Kim et al. 2015, Chen et al. 2017, Cui et al. 2018, Wilder et al. 2018). Aral et al. (2013), Nejad et al. (2015), and Nejad and Amini (2024) explore the role of consumer homophily on the effectiveness of seeding campaigns. Dou et al. (2013) look at how seeding and social media features can be used in tandem to engineer optimal network effects in markets for digital goods and services. Niculescu and Wu (2014) find that uniform seeding dominates feature-limited freemium and no-promotion strategies when consumers significantly underestimate a priori the value of the product. Lin et al. (2019) show that free sampling promotions (including seeding[5]) can have positive effects on product ratings - in other words, seeding can be an effective tool to harvest positive reviews early on in the diffusion process. Han et al. (2021) explore scenarios in which seeding is a desirable strategy for either manufacturer or retailer in a supply chain. Interestingly, recent studies by Chin et al. (2022) and Akbarpour et al. (2023) suggest that targeted seeding might not be more effective, and even possibly less effective, than random seeding. Cui et al. (2024) show that the performance of targeted seeding vs. random seeding depends on the consumers' propensity to spread negative WOM.

[5]Lin et al. (2019) classify the products into nine categories. Not all free sampling campaigns fall under our description of seeding. For example, free samples of health food items do not correspond to our definition of seeding because food is a repeated consumption non-durable item and the sample provides only a small portion. On the other hand, free sampling promotions of apparel and home appliances do correspond to our definition of seeding because these goods are semi-durable or durable, with the same item not being purchased very often.

Surprisingly, despite the widespread use of free trials in software markets, the direct comparison of optimal seeding against TLF, as go-to-market strategies capitalizing on free consumption, remains largely unexplored. Schlereth et al. (2013) conduct a numerical optimization of the market coverage of seeding and TLF under exogenous pricing that is kept constant across the sampling methods. They do not draw conclusions as to which strategy dominates in any given parameter range and they do not benchmark these two business models with free consumption against other no-promotion models in terms of profits. To the best of our knowledge, our study is the first to compare and contrast S, TLF, and business models with no promotion (under both perpetual and subscription-based licensing) within a unified framework accounting for WOM effects, endogenous pricing, adoption costs, and individual use-based value depreciation. Building on the learning framework from Niculescu and Wu (2014), we show that the optimality of seeding previously identified in that study (in the scenarios in which consumers initially underestimate the value of the product) vanishes once free trials enter the strategy set, regardless of the strength of network effects or imperfect learning. Nevertheless, not all is lost for seeding, as we uncover two factors - adoption costs and individual depreciation - that, when accounted for in the model, can restore uniform seeding as the dominant strategy within certain regions of low consumer priors, even when free trials are an option. Collectively, our study contributes both modeling and theoretical advances to the understanding of non-targeted seeding optimality in software markets. Other secondary theoretical contributions (some relegated to the E-companion) include explorations of how the optimality regions fluctuate for various strategies contingent on model parameters.

3 Baseline Model

3.1 Supply Structure and Candidate Business Models

We consider a scenario in which a firm has already developed a software product and is exploring the most profitable way to commercialize it. At this pre-release stage, all the development costs are sunk. In the main setup, we consider a product that has a life span of two periods, after which it becomes obsolete. We show in Section 6.3 that our findings remain robust in the context of a longer horizon as well. The marginal production cost and the time discount factor of future earnings are considered negligible. The firm aims to maximize the undiscounted profit over two periods. Consistent with established literature (Choudhary 2007, Zhang and Seidmann 2010, Niculescu and Wu 2014, Li and Jain 2016, Chen and Jiang 2021), we focus on scenarios where the firm can offer a credible price commitment. In our setup, the firm considers among the following four models:

(a) **Charge for Everything - Perpetual Licensing** (*CE-PL*): Consumers pay a *one-time* fee at the time of adoption, which in turn grants them the right to use the product throughout its remaining life (i.e., until its obsolescence horizon) without any additional charges.

(b) **Charge for Everything - Subscription** (*CE-SUB*): Consumers purchase a single-period license at the beginning of period 1 and/or 2, which expires at the end of that period. Consumers who subscribed in period 1 have the option to renew the subscription at the beginning of period 2 (but are not required to).

(c) **Time-Limited Freemium** (*TLF*): All consumers have access to the product at no charge in period 1 (i.e., the free trial period). When the free trial expires, consumers are required to purchase a license in period 2 to continue using the product. In the context of two periods, only one period is left at the end of the free trial - thus, it does not matter if the *paid* license for period 2 is in a perpetual or subscription-based format. Nevertheless, in Section 6.3, in the context of a longer product lifespan, we do distinguish between the two types of paid licenses at the end of the free trial period.

(d) **Seeding** (S) - paired with **perpetual licensing**: We consider uniform, non-targeted seeding, whereby the firm seeds a fraction k of the customer population uniformly across all tiers of the addressable market (Libai et al. 2005, Li et al. 2019, Chin et al. 2022, Akbarpour et al. 2023).

A key distinction between *TLF* and S lies in how each strategy cannibalizes demand to stimulate paid consumption. *TLF* cannibalizes demand from *every* potential customer for a limited period of time, leaving open the possibility to later charge each of these customers (whose priors have been updated after the free trial) for the residual value of the product after the expiration of the trial. S, by contrast, cannibalizes demand from a subset of the market, albeit for the *entire* product life, relying on those seeded customers to influence the purchase decisions of *other* customers. Accordingly, *TLF* primarily drives paid consumption through self-learning, whereas S leverages self-learning to spark WOM (which in turn, drives social learning).

3.2 Demand Structure and Valuation Learning Process

Consider a unit mass of consumers with their types θ uniformly distributed on $[0, 1]$. A type-θ consumer derives per-period benefits $a\theta$ from using the product. Coefficient $a > 0$, which we hereafter refer to as *quality factor*, quantifies in an aggregate form core quality dimensions of the product such as reliability, versatility, efficiency, ease of use, etc. Type θ captures heterogeneity in the consumers' WTP for quality per period as a reflection of diverse needs for the product and individual fit. Consumers do not observe the true product quality a before the product is released. At the beginning of period 1, *prior to any paid or free adoption in the market*, consumers enter the market with prior $a_1 = \alpha a$ on product quality. In the baseline model, we assume a homogeneous value $\alpha > 0$ across consumers.[6] We relax this assumption in E-companion G and explore how

[6] If $0 < \alpha < 1$ then all customers initially underestimate the quality of the product, whereas if $\alpha > 1$ then all customers initially overestimate the quality of the product.

heterogeneity of consumer priors on quality (whereby some customers initially overestimate the quality of the product while others underestimate it) impacts the main results.

Consumers adjust their priors over time based on learning. Let a_2 denote the consumer's perceived valuation factor at the beginning of period 2. For each period, we assume that any adoption outcome (paid, seeded, or free trial) happens at the beginning of the period whereas learning happens afterwards, throughout the period. Hence, any customer considering whether or not to purchase a license during a given period will act on their valuation priors at the beginning of that respective period. We employ the valuation learning process from Niculescu and Wu (2014), capturing in a unified framework how the value of a_2 is shaped up by *self-learning via use* and *social learning through WOM*, as follows:

- **Self-learning.** We assume that *adopting* consumers (whether paying, seeded, or trying the product) can perfectly learn the product quality through one period of use. We relax this assumption in Section 6.3 in the context of imperfect learning and show that main insights continue to hold. As most software products are experience goods, adopting consumers can *directly* update their priors through their own hands-on experience, which is not necessarily affected by the opinions of others.
- **Social learning via WOM**. Non-adopters in period 1 (for all models except *TLF*), while deprived of direct, own experience with the product, *indirectly* adjust their priors on quality by learning from the "buzz" (WOM) spread by the period 1 adopters. This takes place at the end of the first period, *after* adopters self-learned the valuation of the product and started sharing their signals with non-adopters. We employ the exact same parameterization of social learning as in Niculescu and Wu (2014), whereby non-adopters in period 1, after social learning, enter period 2 with the following updated priors:

$$a_2 = a_1 + N_{1,total}^{\frac{1}{w}}(a - a_1) = a_1\big(1 - N_{1,total}^{\frac{1}{w}}\big) + aN_{1,total}^{\frac{1}{w}}, \tag{1}$$

  where $N_{1,total}$ is the total number of period 1 adopters (including both paying and non-paying adopters, if any)[7] and w is the strength (i.e., the degree of persuasiveness) of the WOM effects.

We refer readers to Niculescu and Wu (2014) for an elaborate discussion of how this WOM-based social learning model is anchored into and motivated by the rich research streams on (i) factors that affect the magnitude of the impact of outside signals and (ii) the stickiness/inertia of own beliefs and strategies in the presence of additional information suggesting a potential need for course correction. In a nutshell, for period 1 non-adopters, the updated prior a_2 at the beginning

[7]Under *CE-PL* and *CE-SUB*, $N_{1,total}$ represents the total amount of *paying* customers in period 1. Under *S*, $N_{1,total}$ includes both *paying* and *seeded* period 1 consumers. Under *TLF*, in the absence of adoption costs, $N_{1,total} = 1$.

Table 1: Consumer perceived quality factor and learning across business models

	Priors at beginning of period 1	Learning during period 1 (after adoption takes place)	Updated priors (posteriors) at beginning of period 2
CE-PL ***CE-SUBS***	All consumers: $a_1 = \alpha a$	Period 1 installed base (paying or seeded): • First, engage in self-learning • Second, after self-learning, at the end of period 1, disseminate WOM Period 1 non-adopters: • Receive WOM signals at the end of period 1 and engage in social learning	Period 1 installed base (paying or seeded): $a_2 = a$ Period 1 non-adopters: $a_2 = a_1 + N_{1,total}^{\frac{1}{w}}(a - a_1)$
TLF	All consumers: $a_1 = \alpha a$	All consumers engage in self-learning	All consumers: $a_2 = a$

of period 2 is a weighted average between the older prior a_1 at the beginning of period 1 and the signal a sent by period 1 adopters after they had experienced the product. The weight of the new signal ($N_{1,total}^{\frac{1}{w}}$) captures the overall impact of WOM effects in convincing non-adopters to *deviate* from their prior beliefs. This impact is shaped by two forces: (i) the volume of outside opinions ($N_{1,total}$), and (ii) the degree of persuasiveness, w, of these signals. In a more general context of combining priors with outside signals, Bates and Granger (1969) show that the minimum variance unbiased estimator for the updated forecast is a weighted average of the prior and the outside signals. Building on that, Zhang et al. (2021) further explain how the resulting weight of the prior in the updated forecast is decreasing in the number of available outside signals. The social learning in this paper, albeit in a reduced heuristic form, remains true to the essence of these theories.

If there are any non-adopters in period 1, then $N_{1,total} < 1$. As such, $N_{1,total}^{\frac{1}{w}}$ is increasing in w, spanning the interval $(0, 1)$ as w spans $(0, \infty)$. A very low w means that outside signals, even in large numbers, have limited power in convincing non-adopters to deviate from their priors. A high w, on the other hand, means that it takes only a handful of outside signals for the non-adopters to adjust from a_1 to a value very close to the real quality factor a. In scenarios in which all consumers either paid for or got free access to the product in period 1 (which is the case under *TLF* model), social learning becomes redundant but remains mathematically consistent with self-learning.[8]

We summarize the two learning mechanisms in Table 1, and our overall key notation in Table A1 in E-companion A. Consistent with Niculescu and Wu (2014), we make several additional

[8] In such a case, $N_{1,total} = 1$ and $a_1 = a$ (via self-learning). As such, regardless of the value of $w > 0$, $a_2 = a + 1 \times (a - a) = a = a_1$. Once customers learn the true quality of the product through self-learning, any subsequent WOM effects have no further impact on their perception of the product quality.

assumptions. On one hand, while each customer knows her own type, the distribution of θ is not publicly known among consumers, such that they cannot infer the true quality a based on the firm's optimal pricing p. On the other hand, the firm knows the consumer type distribution but does not have information on the precise type of each individual customer and can neither price discriminate nor engage in targeted seeding. Moreover, we assume a form of bounded rationality in that consumers in period 1 do not anticipate a change in their priors at a later time (they operate under the belief that their prior is the correct value of quality, especially since they do not know the distribution of θ and they cannot anticipate various scenarios of how demand will be realized).

Without any loss of generality, we normalize the true quality factor to $a = 1$. Moreover, the main results are derived under moderate strength of WOM effects ($w = 1$). We relax this assumption and explore numerically in Section 6.2 how the results hold under varying strengths of WOM effects.

3.3 Dominant Strategy

The individual equilibrium solutions for each of the four strategies under the baseline setup are presented in E-companion B, in Propositions B.1-B.4. We point out that the optimal pricing solutions under *CE-PL* and *S* (in Propositions B.1 and B.4) are reproduced directly from Niculescu and Wu (2014). Note that *CE-PL* is essentially a special case of model *S* with the seeding ratio k set to zero. For clarity of exposition, for a given parameter set, we consider *CE-PL* to dominate *S* if the profit optimization under model *S* yields $k^* = 0$. Below, we present the dominant strategy, when comparing among the considered four models, for each region of the parameter space.

Proposition 1. ***[Dominant Strategy in Baseline Model]*** *Under the baseline setup, there exists* $\bar{\alpha} \in (0, 1)$[9] *such that the firm's dominant strategy is:*

(i) TLF, if $\alpha \in (0, \bar{\alpha})$*;*

(ii) CE-SUB, if $\alpha \in [\bar{\alpha}, 1)$*;*

(iii) CE-PL, if $\alpha \geq 1$.

In addition, TLF yields the highest social welfare among the four models.

All proofs are included in the E-companion. Figure 1 illustrates the firm's optimal price and ensuing adoption pattern and profit under each of the four strategies. Note that, in panel (b), under *CE-SUB* the price is per period, whereas for the other go-to-market strategies the price is for the perpetual license. Panels (c) and (d) capture the adopters paying for *new* licenses in periods 1 and 2, N_1^* and N_2^*. We point out that N_1 is different from $N_{1,total}$ (from equation (1)) as the latter includes all adopters (paying/seeded/free trial). In period 2, there is a nuanced distinction between "new consumers buying licenses" and "consumers buying new licenses." In the context of *CE-PL*

[9] $\bar{\alpha} \approx 0.3968$ is defined in implicit form in the proof of Proposition 1.

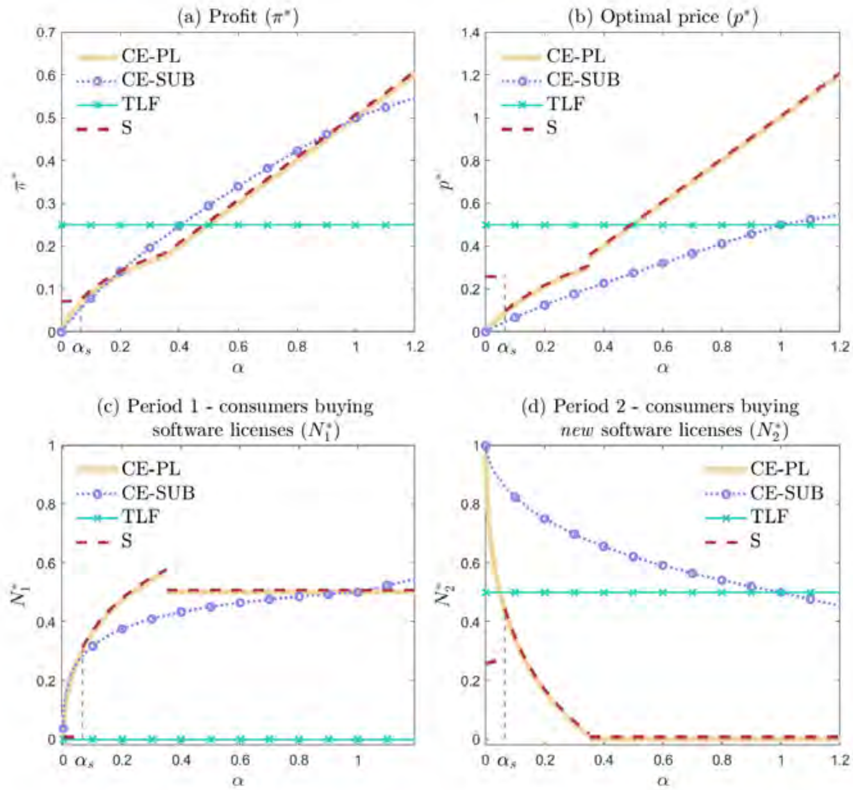

Figure 1: Baseline setup - comparison of go-to-market strategies.

and S, N_2 effectively captures new market entrants (since period 1 adopters are grandfathered into period 2 by the perpetual license). However, for *CE-SUB*, N_2 includes, in addition to new first-time subscribers (if any), also period 1 adopters that decide to renew the subscription, as they essentially purchase a *new* single-period subscription for period 2. For TLF, N_2 represents the subset of free trial consumers that decide to continue with paid use at the end of the promotion.

An immediate implication from Proposition 1 is that seeding a non-negligible mass of consumers upfront is never optimal, as can be seen in panel (a) of the figure. When optimizing firm strategy under S *in isolation*, $k^* > 0$ is optimal only when consumers initially severely underestimate the quality of the product ($\alpha \in (0, \alpha_s)$ with threshold $\alpha_s \approx 0.065$ defined in Proposition B.4). For $\alpha > \alpha_s$, S defaults to *CE-PL* - this can be seen in Figure 1 as the plots for the two strategies overlap in this α-range.[10] It can be shown (from Propositions B.1, B.2, and B.4 in E-companion B) that $\lim_{\alpha \downarrow 0} \pi^*_{CE-PL} = \lim_{\alpha \downarrow 0} \pi^*_{CE-SUB} = 0 < \frac{1}{16} = \lim_{\alpha \downarrow 0} \pi^*_S$. Niculescu and Wu (2014) discuss in detail the mechanics of how S dominates *CE-PL* under low priors and a similar argument applies to the dominance of S over *CE-SUB* in the same region. In essence, when consumer priors on

[10]The discontinuity in price and period 1 paying adopter base for *CE-PL* and S at $\alpha = 13 - 4\sqrt{10} \approx 0.35$ captures the shift in the pricing strategy towards abandoning period 2 adoption (hence abandoning pursuit of WOM effects) and maximizing revenue extraction exclusively from period 1 adoption by higher-type consumers, once their priors are significant enough. This shift is discussed in detail in Niculescu and Wu (2014).

quality are low, in the absence of a free offering that would facilitate self and social learning, the firm would have to rely on a very low price in order to jumpstart adoption, thus taking in only a small profit. In contrast, in that same region, under S, WOM effects do not need to be triggered by paid adoption. Instead, the firm can forfeit period 1 paid adoption altogether, and use seeded customers to ignite social learning that can induce a considerable update of the priors of unseeded customers (such that a significant number of the latter are willing to pay a substantial price in period 2 for a *single* remaining period of use even though they balked at paying the very same price for two periods of use at the beginning of period 1). Thus, without TLF among available options, S would emerge as the dominant strategy in this region of the parameter space.

However, in contrast to Niculescu and Wu (2014), what we find is that when TLF enters the picture, that aforementioned region of optimality for S evaporates. Under optimal implementation of S and, by definition, under *TLF*, for low enough priors, the firm does not get any paid adoption in period 1. What ultimately decides the winner between these two strategies is the revenue in period 2. Under S, the seeded customers steer other customers in the direction of the right value of the quality factor. Nevertheless, as seeded customers get perpetual licenses and seeding is uniform, the firm cannot avoid seeding a fraction of the higher valuation population - however, the firm cannot afford seeding too many of the high type customers that could be payers in period 2 (under updated priors). Also, unseeded customers do not update their prior all the way to the correct value of the quality (one would need complete market seeding, i.e., $k = 1$, for that, which would essentially erase all profits). However, under *TLF*, during period 1 trial, all customers update their priors all the way to the correct value of the quality factor. Moreover, the trial version is not offered under perpetual license - *all* customers remain in the pool of potential paying adopters in period 2. As such, the firm can collect revenue in period 2 from more high type customers, also with higher updated WTP, under TLF compared to S. Hence, TLF dominates S in that region.

The dominance of TLF extends well beyond α_S all the way to $\bar{\alpha} \approx 0.41$. While both *CE-PL* and *CE-SUB* strategies become progressively more profitable with higher α, under each of these strategies the firm is still considerably constrained by the priors and cannot price too high upfront. While both of these strategies rely gradually less and less on *new* period 2 adopters (which have not adopted in period 1) as consumer priors increase, the firm gives up on this component of revenue, and essentially on engendering social learning, considerably faster under *CE-PL* than under *CE-SUB*, as can be seen in panel (d) of Figure 1. Under *CE-PL*, social learning would have to yield more than a doubling of the priors for a period 1 non-adopter to even consider period 2 adoption (since they are now looking at only one remaining period of use before obsolescence and they balked at the same price for a 2-period perpetual license before). As α increases, under *CE-PL*, for the firm to ensure such strong WOM thrust for period 2 new adoption to occur, it would have to induce enough period 1 adoption, which would put downward pressure on the price it charges and be suboptimal beyond a certain point. However, under *CE-SUB*, as α increases, the firm continues to make use of *both* self and social learning as long as $\alpha < 1$ (all period 1 adopters renew subscription in period 2 and also new adopters enter the market in period 2, i.e., $N^*_{2,CE-SUB} > N^*_{1,CE-SUB}$ for all $\alpha \in (0,1)$). The difference is because period 1 non-adopters under *CE-SUB*, relative to *CE-PL*, initially balked at a 1-period subscription instead of a 2-period perpetual license - thus, WOM effects do not have to be that strong to induce new adopters to enter in period 2. This added flexibility allows *CE-SUB* to overtake *CE-PL* when α gets above roughly 0.17.

It is this same flexibility that eventually enables *CE-SUB* to flip the tables and dominate *TLF* once customers only moderately or slightly underestimate the initial value of the product ($\alpha \in [\bar{\alpha}, 1)$). Under *TLF* it is optimal to have precisely half of the population paying for adoption for the second period. As α increases in this region, under *CE-SUB*, we see from panels (c) and (d) of Figure 1 that the firm will optimally induce a little less than half of the population to pay for adoption in period 1 and a little more than half of the population to pay for adoption in period 2 (with $N^*_{1,CE-SUB} + N^*_{2,CE-SUB} > 1$ for all $\alpha \in [\bar{\alpha}, 1)$). With higher priors, the firm is able to charge a high enough per-period subscription price (for most of this region we have $p^*_{CE-SUB} \geq p^*_{TLF}/2$), which ensures that from two periods it will collect more revenue than under *TLF*.

If consumers initially overestimate the product ($\alpha \geq 1$), then *CE-PL* strategy has the upper hand as it relies only on period 1 adoption, charging consumers for two periods *before* they get a chance to update their priors (downwards) through learning. On the other hand, both *TLF* and *CE-SUB* are impacted by consumer valuation learning, which (under either self-learning or social learning) leads to a downward calibration of priors and, implicitly, of the consumers' WTP.[11]

In terms of social welfare, *TLF* dominates. Since price represents an internal transfer, with development costs sunk and negligible marginal costs, residual social welfare amounts to consumer surplus. Under *TLF*, all customers get to use the product in period 1, and the top half of them (in terms of valuation) pay for it also in period 2. None of the other models achieve an aggregate product use similar to *TLF*. What this translates to is that the firm will choose a socially optimal strategy only when consumer priors are low (i.e., $\alpha \in (0, \bar{\alpha})$).

4 Model with Individual Depreciation

In the baseline scenario, we assumed that consumers can extract value from the product at the same rate for as long as they use it. However, as discussed in the Introduction, certain products exhibit

[11]While under *CE-SUB* the firm gets some paid adoption in period 1 (from selling one-period subscriptions), all those subscribers will downgrade their priors via self-learning and also WOM will push downwards the priors of the non-adopters in period 1. Hence, under *CE-SUB* the firm would face churn in period 2, without any new adopters.

individual depreciation, whereby consumer satiation, diverted interest, or a limited need can lead to reduced product valuation *past first (initial) period of use* (Dou et al. 2017, Tan 2024). Han et al. (2016) showed that individual depreciation occurred widely in the intertemporal use of information goods and services. They found that, among mobile apps, the satiation level was highest for portal search apps (which users tend to use briefly, for quick searches), and lowest for communication apps (which consumers use at a sustained level to interact with others). According to Statista (2025), consumer interest is shortlived for many installed apps, with retention rates plunging to single digits after only a month. Zooming in on the video game industry, studies by Shiller (2013) and Ishihara and Ching (2019) further corroborate the presence of individual depreciation.

To capture this effect, we propose an adjustment to our baseline model. More specifically, for period 1 adopters, the value they can extract from period 2 scales by a factor $\lambda \in (0, 1]$.[12] On the other hand, period 1 non-adopters are not affected by individual depreciation (their perceived period 2 valuation can only be impacted by WOM effects). In other words, period 1 usage cannibalizes period 2 benefits. We assume that λ is common knowledge.[13] In this section, we consider λ to be exogenously determined. Thus, we look exclusively at the post-development go-to-market strategy, considering already sunk the costs for any (additional) features and content that can impact depreciation. We relax this assumption in Section 6.1, endogenizing λ in the context of a more general model, and show that the main insights continue to hold qualitatively. The separate equilibrium solutions for each of the four strategies under the individual depreciation scenario are presented in E-companion C, in Propositions C.1-C.4. Below, we characterize the dominant strategy, when comparing among the four considered strategies, for each region of the parameter space. We point out that the individual equilibria and the comparison of strategies are highly nontrivial, with the depth of the analysis included in the proofs.

Proposition 2. *In the presence of individual depreciation, the firm's dominant strategy, as illustrated in Figure 2, is:*

(i) CE-PL, if $0 < \alpha_1(\lambda) < \alpha$ *(yellow region);*

(ii) TLF, $\lambda_t < \lambda \leq 1$ *and* $0 < \alpha < \alpha_2(\lambda) < 1$ *(green region);*

(iii) Otherwise,

(a) CE-SUB, if $\alpha > \alpha_t$ *and* $\lambda > \lambda_x(\alpha)$ *(purple region);*

(b) S, otherwise (red region).

[12] For example, under *TLF*, since all customers get to try the product for free for one period, the perceived period 2 utility becomes $u_2 = a_2\theta\lambda - p$. Similarly, under *CE-PL*, the perceived utility at the beginning of period 1, before release, is $u_1 = a_1\theta(1 + \lambda) - p$ (which is the same case for unseeded consumers under *S*). At the beginning of period 2, period 1 non-adopters exhibit utility $u_2 = a_2\theta - p$.

[13] Firms do market research about the use cases for their products. On the other hand, consumers, even without fully understanding a priori the quality of the product, know their own needs and preferences for features and content.

Functions $\alpha_1(\lambda)$*,* $\alpha_2(\lambda)$*,* $\lambda_x(\alpha)$*, and exogenous thresholds* α_t *and* λ_t *are defined in E-companion C and marked on Figure 2. In addition,* TLF *yields the highest social welfare among the four models.*

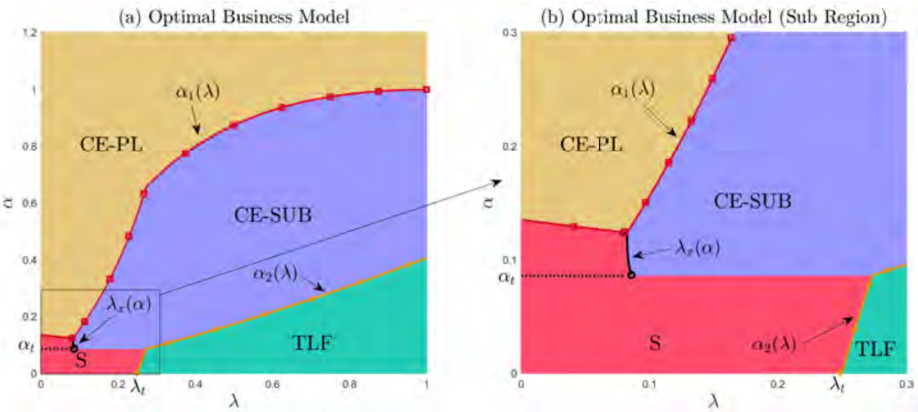

Figure 2: Setup with individual depreciation - optimal strategies. Panel (b) represents a zoomed-in snapshot of the left-bottom corner region from panel (a).

Comparing Propositions 1 and 2, we see that individual depreciation plays a non-trivial role in determining which strategy is dominant in various regions of the parameter space. We recognize Proposition 1 results as the vertical slice at $\lambda = 1$ in panel (a) of Figure 2. In contrast to the baseline setting, in the presence of individual depreciation, *a major difference is that S can emerge as the optimal strategy* - that happens in scenarios involving concomitantly severe individual depreciation through use and significant prior underestimation of the product value (both λ and α small enough). In this region, offering the product for free to a fraction of the population leads to a significant update in the product valuation for the rest of the consumers. This allows the firm to charge a substantial price under S, with paid adoption taking place in second period only, following the realization of WOM effects, thus leaving unseeded population largely unaffected by individual depreciation. Under strategies without a free offering (*CE-PL* and *CE-SUB*), the firm would have no option but to price low from the beginning, trapped by the inability to generate WOM from other sources rather than the paying installed base. At the same time, with strong individual depreciation (small λ), TLF is no longer optimal regardless of consumer priors because all consumers extract the bulk of the product value during the free trial and there is little residual value to be extracted in period 2, leading to very low WTP. Also, in the same strong individual depreciation region, if the initial valuation priors are not too low ($\alpha > \alpha_1(\lambda)$), *CE-PL* still outperforms S because the WOM effects generated from the seeded consumers cannot shift the dial on valuation by too much to justify forfeiting revenue from the high-type seeds if the initial estimation is not too low (*CE-PL* can be offered with a high price from the get-go).

But there is more nuance to the contrast between S and TLF in the presence of depreciation. It is not only that S can emerge as optimal but also that, in fact, each of the four considered strategies has an optimality region. Similar to reasoning under the baseline case, low priors support strategies with some free consumption component that ignites WOM effects (S or TLF) while high priors support strategies with no free consumption, that rely less on WOM effects (*CE-SUB* and *CE-PL*). As discussed above, when λ is low, S is the one strategy with free offerings that strongly competes with *CE-PL* or *CE-SUB*, whereas when λ is high, it is TLF that dukes it out with the non-free strategies. In the low λ region, the boundary between S and the non-free strategies is weakly *decreasing* in lambda - S is gradually losing ground when there is less depreciation. At the same time, for higher λ, the boundary between TLF and the non-free strategies, $\alpha_2(\lambda)$, is *increasing* in λ - TLF is gaining ground when there is less depreciation. Consumer prior α is irrelevant to TLF with or without depreciation. All consumers learn the true value of the product via the free trial. In contrast, for S to dominate, it is important that it induces paid adoption solely in period 2 (with higher price, no period 1 paid adoption, and WOM effects carried through only by the seeded customers in period 1); hence, when S dominates, λ is irrelevant to it (given that paying customers are not exposed to the product in period 1).

When λ is very low, as it increases, under *CE-PL* and *CE-SUB* the firm can effectively increase the prices. As such, in this range, the boundary between *CE-PL* / *CE-SUB* and S will decrease because S needs gradually lower priors (that keep the profitability of the other strategies in check in spite of less depreciation) to still dominate. As depreciation becomes less severe ($\lambda > \alpha$), but still relatively low, under *CE-SUB* all period 1 consumers return in period 2 (in addition to new consumers joining in period 2). In this region, the update in valuation due to WOM dominates depreciation such that π^*_{CE-SUB} is independent from λ altogether (as can be seen in region (a) of Proposition C.2 in E-companion C). Hence, λ does not affect either *CE-SUB* or S in this case and the boundary between the two optimality regions is flat at α_t. However, as we get into intermediate to large values of λ (even less depreciation), TLF dominates *S*. In such ranges, as individual depreciation lessens, TLF becomes more profitable due to an increased consumer WTP. Nevertheless, the profit under *CE-SUB* does not change w.r.t. λ as the boundary between *CE-SUB* and TLF ($\alpha_2(\lambda)$) remains inside the region $\lambda > \alpha$. Thus, as λ increases, TLF gradually gains more advantage over *CE-SUB* for low to moderate α.

5 Model with Adoption Costs

In this section, we extend our baseline model by accounting for a *one-time* adoption cost $c \geq 0$ incurred during the period of the *initial* adoption (whether paid or for free).[14] Under TLF and *CE-*

[14]Under *CE-PL*, at the beginning of period 1, consumers perceive utility $u_1 = 2a_1\theta - c - p$. Period 1 non-adopters, after updating their priors, face perceived utility $u_2 = a_2\theta - c - p$ at the beginning of period 2. Under TLF, at the

SUB, adopters from period 1 do not incur the adoption cost again, even though they make another adoption decision at the beginning of period 2. For simplicity, in this setup, we assume no depreciation (but will later numerically explore a generalized model with both depreciation and adoption costs in Section 6.1). In the software industry, installation and configuration processes have become increasingly complex and resource-demanding for highly specialized apps. For instance, a complete Windows installation of Matlab R2025a requires 24 GB of storage, roughly 6 times the amount required for Matlab 2014a.[15] Additionally, modern enterprise software often involves initial migration costs, legacy integration, extensive customization, and potentially organizational change. On the other hand, adoption costs trend lower when using an app with minimal on-premise resource requirements, intuitive interface, and confined scope (e.g., cloud-based collaboration and productivity apps such as Google Workspace, Zoom, Slack). While individual depreciation impacts only consumers that use the product for two periods, adoption costs affect every user.

A major difference from prior scenarios is that, when accounting for adoption costs, under S and TLF, some of the customers presented with the free offering *choose to decline it*. In addition, WOM effects start playing a role for *TLF* because only the customers that choose to use the product during the free trial perfectly learn their valuation. As such, these customers will generate WOM effects for all the other customers that did not enroll in the free trial. The separate nontrivial equilibrium solutions for each of the four strategies under the adoption costs scenario are presented in E-companion D, in Propositions D.1-D.4. The following proposition fully characterizes the firm's dominant strategies across the entire parameter space.

Proposition 3. *In the presence of adoption costs, the firm's dominant strategy, as illustrated in Figure 3, is:*

(i) If $\alpha \leq \frac{c}{2}$*, don't enter the market, (white region);*

(ii) If $\alpha > \frac{c}{2}$*, enter the market with:*

- *(a) CE-SUB, if* $\hat{\alpha}_2(c) < \alpha < \hat{\alpha}_1(c)$ *(purple region);*
- *(b) TLF,* $\hat{\alpha}_3(c) < \alpha < \hat{\alpha}_2(c)$ *(green region);*
- *(c) Otherwise,*
  - *(1) S, if* $\alpha < \hat{\alpha}^{\dagger}(c)$ *and* $c < c^{\ddagger}(\alpha)$ *(red region);*

beginning of period 1, customers consider the utility from the one-period free trial $u_1 = a_1\theta - c$. Trial participants in period 1 anticipate $u_2 = a_2\theta - p$ at the beginning of period 2, whereas the folks that declined the free trial in period 1 anticipate $u_2 = a_2\theta - c - p$. Under S, unseeded customers face the same utilities as under *CE-PL*. Customers presented with a seed face perceived utility $u_1 = 2a_1\theta - c$ over the offered two-period perpetual license. Seeded customers that declined the seed offer in period 1 anticipate $u_2 = a_2\theta - c - p$ in period 2. Under *CE-SUB*, customers face perceived utility $u_1 = a_1\theta - c - p$ in period 1. Installed base from period 1 anticipate $u_2 = a_2\theta - p$ in period 2, whereas consumers that did not subscribe in period 1 anticipate $u_2 = a_2\theta - c - p$.

[15] `https://www.mathworks.com/support/requirements/previous-releases.html`.

(2) CE-PL, otherwise (yellow region).

Functions $\hat{\alpha}_1(c)$, $\hat{\alpha}_2(c)$, $\hat{\alpha}_3(c)$, $\hat{\alpha}_{\dagger}(c)$, and $c^{\ddagger}(\alpha)$ are defined in E-companion D and marked on Figure 3. Notably, TLF yields the highest social welfare among all models.

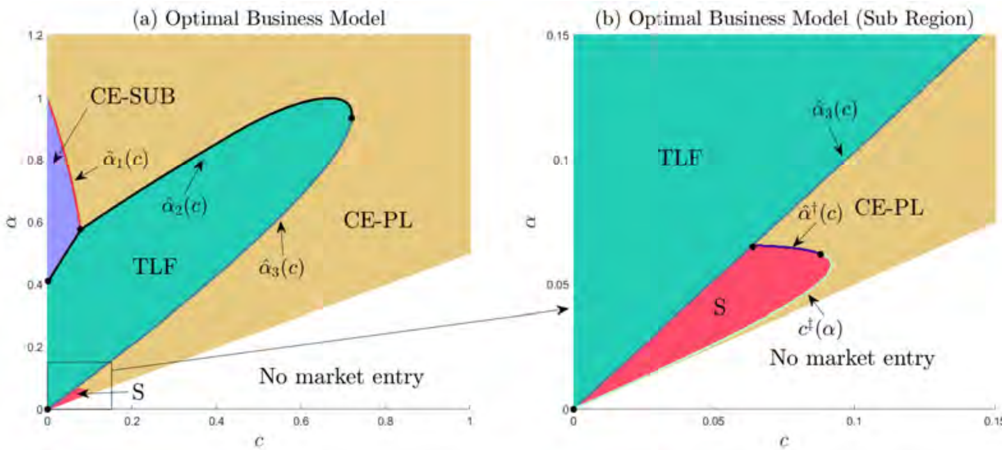

Figure 3: Optimal strategies - model with adoption costs. Panel (b) represents a zoomed-in snapshot of the left-bottom corner region from panel (a).

When adoption cost is too high ($c \geq 2\alpha$), no user will adopt the product regardless of the business strategy employed (including those with a free offering). Hence, the firm cannot profitably enter the market in this region. Conversely, when adoption costs are manageable ($c < 2\alpha$), each business strategy can emerge as optimal. When $c = 0$, we recognize the market outcome from the baseline model (vertical slice in panel (a) of Figure 3 at $c = 0$). It is important to mention that when $c = 0$ and α is very small, TLF strictly dominates S.

However, as c increases, S *emerges as optimal* as long as c and α are sufficiently small (as illustrated in part ii.c.1 of the above result). In particular, the region where S dominates intersects (and extends upwards in α slightly beyond) the range $\alpha < c < 2\alpha$, which is a range not contested at all by TLF, as the latter can only be profitably implemented under stricter constraint $c < \alpha$.[16] Interestingly, when c is very close to 2α (right above the no-market-entry threshold), *CE-PL* will dominate S - only the very top-valuation seeded consumers are willing to consider the free offer and, as such, the generated WOM effects would be too weak to significantly boost the valuation of the other consumers (to reap the benefits only through period 2 adoption). Thus, it is more profitable to just price a 2-period perpetual license at a low level and sell it to those top-valuation consumers directly. But if we move further above the market entry threshold (i.e., a slightly larger value of $2\alpha - c$), under S the firm can generate a slightly stronger WOM effect through a larger number

[16] TLF profitability can occur only under $c < \alpha$, as it is necessary to induce at least some enrollment in the one-period free trial in order to ignite WOM (as seen in Proposition D.3 in E-companion D).

seeds, enough to give an edge to S over *CE-PL* (as the latter strategy can only employ a small price). Note also that *CE-SUB* can only dominate for small adoption costs (following dynamics similar to those discussed in Section 3.3, which we skip discussing here for brevity) but higher priors, as it necessitates a jumpstart of adoption in period 1 with a high enough price for only a single period subscription when consumers face both adoption costs and the subscription price for that initial period. As costs increase, it loses to either TLF (due to the free trial customers facing adoption costs but no price in period 1) or *CE-PL* (due to period 1 customers amortizing adoption cost over two periods of the perpetual license).

For intermediate adoption cost c, two forces work against S. First, higher adoption costs shrink the customer pool for seeding (in spite of the free giveaway), which in turn reduces the ability to generate WOM effects. Second, as costs increase, the feasible market-entry region also requires higher priors, so that the firm can monetize the customers more efficiently to make entry worthwhile. Similar to the baseline model dynamics, under higher α, there is less benefit from strong WOM effects and it becomes suboptimal to forfeit revenue from some of the top-valuation customers for both periods. On the other hand, for higher priors $\alpha > c$, TLF is somewhat less affected by the intermediate adoption costs because the firm has the ability to monetize all free trial customers (including all top-valuation ones) in the second period since, for those, the adoption cost will be already sunk and their valuation will be updated to the true value. In this region, *CE-PL* dominates TLF at relatively low and high α values, whereas for intermediate α, the reverse occurs. Again, for $\alpha < c < 2\alpha$, TLF cannot be profitable at all, whereas for large priors it is not optimal to offer anything for free. In the intermediate range of α, TLF gets an edge over *CE-PL* because the jump in valuation via self-learning is still significant and enough customers try the product during trial, allowing the firm to charge a high price in period 2 alone. Under *CE-PL*, the firm has to start at a somewhat lower price and remains committed to it. Once we get into very large c ranges though, TLF loses the advantage across all regions, and *CE-PL* remains the sole dominant strategy.

6 Robustness Checks

We conduct several robustness checks by relaxing various model dimensions. First, in Section 6.1, we account for individual depreciation and adoption costs simultaneously. Next, in Section 6.2, we explore how the strength of WOM effects shapes our results. Lastly, in Section 6.3, we extend our model to 3 periods (hence shortening the length of the free trial relative to the product lifetime) and consider imperfect self-learning (whereby product use does not fully reveal the true valuation within one period), social learning based on multiple outside signal values, and an additional go-to-market strategy (subscription with free trials, alongside perpetual license with free trials). Furthermore, in E-companion G, we consider heterogeneity of consumer priors whereby, initially, some consumers overestimate while others underestimate the true product value. Collectively, these extensions confirm that S can emerge as the optimal strategy in the presence (but not in the absence) of individual depreciation and/or adoption costs in a more general context. Given the complexity of these extended models, closed-form solutions are intractable; hence, we run numerical explorations to identify optimal strategies for each extension.

6.1 Extension 1 - Model with Both Individual Depreciation and Adoption Costs

In this section, we consider a model that includes *both* individual depreciation and adoption costs (combining the frameworks from Sections 4 and 5). Here, consistent with the main setup, we consider an exogenous individual depreciation rate. In E-companion F, we show that our results continue to hold when the firm endogenizes the individual depreciation rate (in which case the firm controls the degree of depreciation through new content/features).

We explore the 3-dimensional parameter space $\{\alpha, c, \lambda\}$ and present in Figure 4 several slices of the outcomes under this parameter space, at three distinct and relatively small α values (0.05, 0.1, and 0.15). Prior insights continue to hold qualitatively. We confirm that all four go-to-market candidate strategies can be optimal, with S dominating in regions of high depreciation (low λ) and low adoption costs. At very low priors, strategies with free offerings are superior, but non-free strategies gradually take the stage as priors start increasing. Note that when $c = 0$ and $\lambda = 1$, we default to the baseline scenario in which S is never optimal.

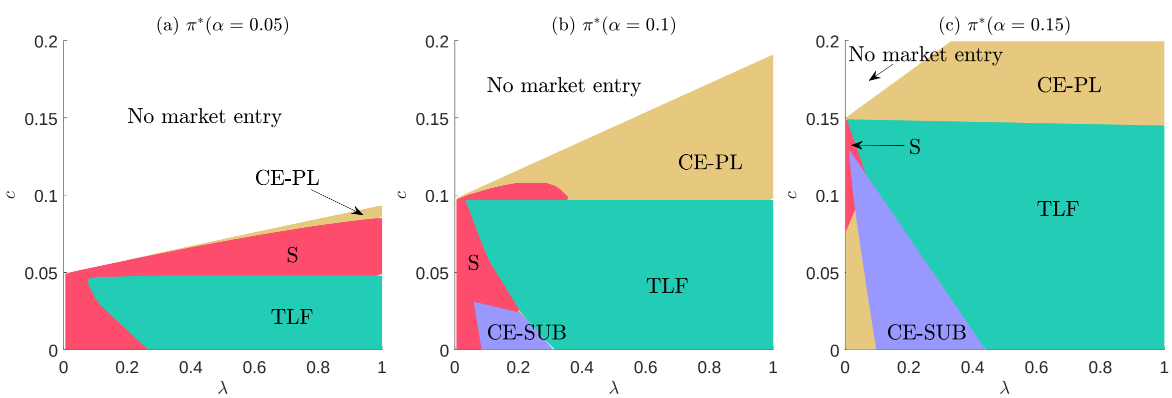

Figure 4: Optimal Strategies - model with adoption costs and exogenous individual depreciation

6.2 Extension 2 - Model with Generalized Social Learning

The main results consider moderate strength of WOM effects ($w = 1$), mainly for tractability purposes. In this section, we relax this assumption in order to explore the robustness of our results in contexts of a wide range of intensities of WOM effects from weak (low w) to strong (high w).

Figure 5 captures how the baseline model behaves under general WOM. We confirm that regardless of the strength of network effects, in the baseline model, in the absence of adoption cost or

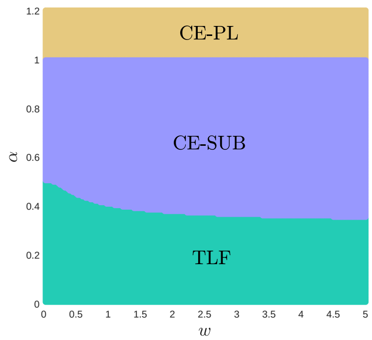

Figure 5: Optimal strategies - baseline model with generalized WOM

individual depreciation, S is always dominated. When $\alpha \geq 1$, *CE-PL* is always the optimal model. When $\alpha = 1$, *CE-PL* and *CE-SUB* perform identically. When $\alpha \leq 1$, *TLF* is not affected by WOM and it always dominates seeding as the latter cannot induce perfect valuation learning via WOM, regardless of strength. However, asymptotically, as $w \to \infty$, we have $\pi_S^* \uparrow \pi_{TLF}^*$ (convergence from below) as it takes a gradually smaller volume of seeding to achieve increasingly better learning. At the same time, we notice that stronger WOM effects also lead to an expansion of the *CE-SUB* optimality region at the expense of the *TLF* as the firm gains more flexibility to price the subscription higher per period, balancing out less period 1 adoption at higher margins with strong WOM effects that lead to more new consumers in period 2 even when the period 1 adoption shrinks.

Our further analysis suggests that, when the individual depreciation and adoption costs are accounted for, the results under general WOM also remain consistent with our prior findings from Sections 4 and 5. This can be seen in Figures 6 and 7. We consider for both explorations a triplet of WOM strength effect values $w \in \{0.5, 1, 2\}$. Serving as benchmark, panel (b) in both of these figures is identical to the corresponding figures in previous sections (Figures 2 and 3). First, we confirm that S will always show up as optimal when consumers underestimate the value of the product (and several other conditions are met). Similar dynamics as the ones discussed before are at play here as well, and, for that reason, we omit such discussions here for brevity.

Even though social learning impacts multiple strategies, an interesting pattern emerges:

Remark 1. *When consumers incur either adoption costs or individual depreciation, stronger WOM effects lead to an expansion of the optimality area for S.*[17]

This insight is in contrast with the baseline model, in which WOM effects never give an edge to S in the four-strategy race. Hence, it is very important for firms to understand *how* to incorporate the magnitude of WOM effects in their decision making process and choice of go-to-market strategies.

[17] While Figures 6 and 7 contain only three values for w we have numerically tested this insight for many others, including significantly larger w values and confirm the validity of the insight.

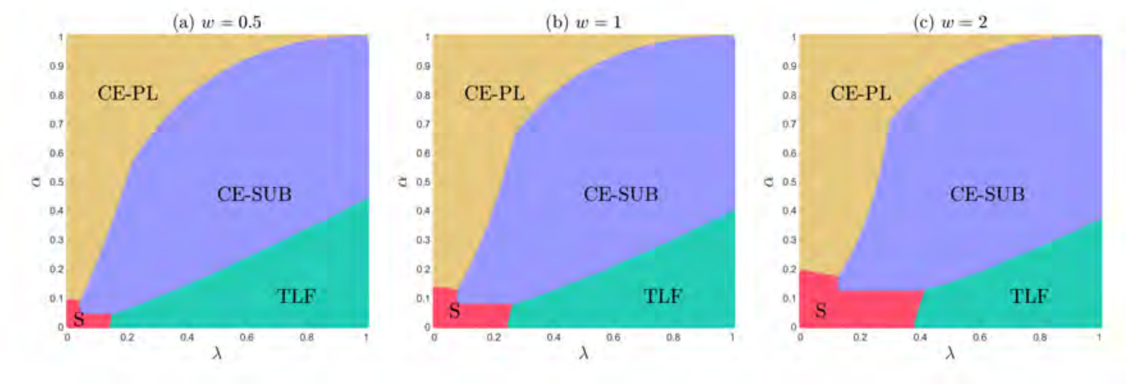

Figure 6: Optimal strategies - model with individual depreciation and generalized WOM

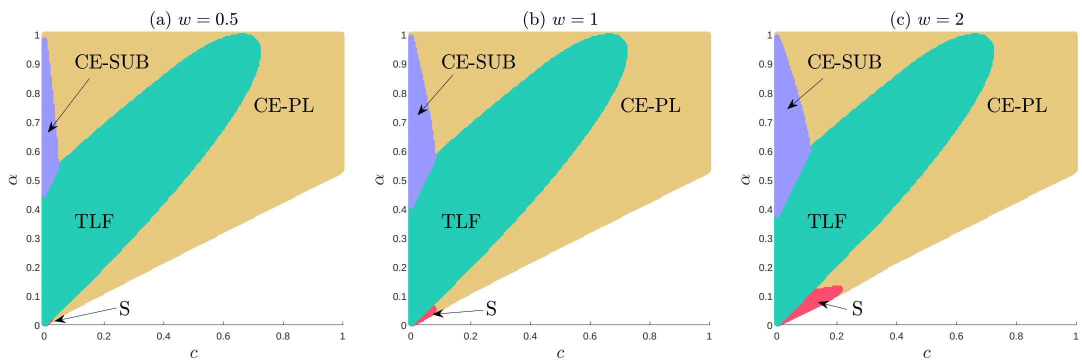

Figure 7: Optimal strategies - model with adoption costs and generalized WOM

More so, stronger WOM effects allow for a more efficient seeding approach with far fewer seeds needed to raise awareness about the true value of the product. At the same time, TLF comes under increasing pressure from *CE-SUB* as the latter is able to capitalize on WOM effects as well.

6.3 Extension 3 - Imperfect Self-Learning with 3 Periods

In this final robustness check, we jointly relax two assumptions in our main model – the 2-period horizon and the perfect 1-period self-learning. First, we extend the analysis to a *3-period setting* (but qualitatively similar insights hold for a larger number of periods as well), making free trials last a smaller share of the product life cycle and thereby improving TLF's monetization potential. A multi-period framework also captures potentially compounding learning cycles, as users continue updating their priors about the product's true value over successive periods, enabling a more granular representation of the learning process. Second, in tandem with the extended horizon, we introduce *imperfect self-learning* – we accommodate not only for the fact that it may take longer for adopters to uncover the true value of the product through self-learning, but also for their learning potentially overshooting in either direction. Thus, adopters update their priors after each period of use but may not converge to the true valuation within a single period.

We confirm that the insights from the main model remain robust under this extension as well. In particular, S is always dominated in the absence of adoption cost or individual depreciation. However, when either adoption costs are present but low or there is high depreciation, S does emerge as the dominant strategy when initially consumers significantly underestimate the value of the product. For brevity, all details are relegated to E-companion E.

7 Impact of Gen/Agentic AI on Strategies with Free Consumption

While this study examines an unconstrained parameter space, the rapid emergence of powerful GenAI tools warrants consideration of how our findings will remain applicable in the evolving technological landscape. The year 2022 marked a pivotal moment in human-computer interaction with the public release of ChatGPT by OpenAI, soon followed by a proliferation of similar GenAI and, more recently, Agentic AI tools from various market participants. This ongoing AI revolution is transforming the software industry and underscores the need for rigorous theoretical reassessment of its implications for a broad spectrum of established business strategies (Korzynski et al. 2023, Hermann and Puntoni 2024, Huang and Rust 2025), *particularly those involving consumer learning.*

Table 2: Landscape shift: impact of GenAI on model primitives

Parameter	c	w	α	λ
Impact	$\downarrow$	$\uparrow$	$\downarrow$ or $\uparrow$	$\downarrow$ or $\uparrow$

Below, in Subsections 7.1-7.4, we describe qualitatively in detail several potential mechanisms through which Gen/Agentic AI reshapes key model primitives, thereby driving markets to specific feasible ranges of the parameter space and influencing the relative dominance of strategies such as S and TLF. As a preamble, we summarize in Table 2 these directional impacts. This discussion remains grounded in the quantitative analysis established in the main text.

In real markets, Gen/Agentic AI will likely impact multiple parameters simultaneously. Given its effects on c and w (discussed below in Subsections 7.1 and 7.2), if the net impact pushes α further away from the real valuation (which, as discussed below in Section 7.3, we expect to happen in the short run), then, when customers initially underestimate the value of the product, *both S and TLF will be brought into sharper focus as increasingly relevant* (with the winning strategy determined based on where λ falls). To our knowledge, this represents one of the first theoretical treatments on how Gen/Agentic AI influences the optimality of free-consumption strategies.

7.1 Impact of Gen/Agentic AI on the Adoption Cost (c)

First, the widespread adoption of GenAI is expected to substantially reduce software users' adoption costs. This stems from GenAI's ability to provide highly personalized configuration and training processes accessible to users on demand. Through chatbots and prompt-based tools, users can accelerate software deployment in complex business environments and markedly enhance productivity (Bick et al. 2025, Brynjolfsson et al. 2025). Furthermore, many software firms now integrate GenAI modules directly into their products, while others enable GenAI connectivity via APIs (Russo 2024) or offer services to customize/train GenAI to intrinsic client needs. For instance, Salesforce's Einstein GPT, linked with OpenAI's models, provides intelligent support during CRM implementation, and Agentforce tools allow users to build and deploy autonomous AI agents operating across business functions. Similarly, Microsoft Azure AI Studio and Google AI Studio empower users to develop AI assistants by combining general-purpose large language models (LLMs) with proprietary data. GenAI has also transformed software development through programming assistants such as Amazon Q Developer, Anthropic Claude Code, GitHub Copilot, and Google's Gemini Code Assist and Jules, which significantly reduce the coding expertise and effort traditionally associated with complex coding tasks. Moreover, the continued evolution of GenAI is expected to further lower barriers to low-code and no-code platforms, indirectly reducing adoption costs for syntax-based programming and profoundly reshaping software development (Ghoshal 2023).

Adoption costs are unlikely to vanish entirely, as software integration and configuration still require effort, resource-intensive applications may necessitate hardware upgrades, and use of advanced GenAI API or specialized AI assistants currently incur small fees. Thus, we expect GenAI to move c to a lower (yet positive) range, which, as shown in Sections 5 and 6, limits the market to a parameter range where *all* strategies, including those with free consumption, can be optimal.

7.2 Impact of Gen/Agentic AI on Social Learning (w)

Secondly, GenAI is likely to improve social learning efficiency, which in our framework corresponds to a higher w. This is because GenAI can train itself on feedback and data from users (and prompt engineers/testers) of a software app early in the adoption cycle, with improved models benefiting the training and education of potential users of that app in the later periods. Once the early adopters start using the app and further train GenAI with respect to it, a prospective user can now query that same AI assistant about how (or whether it is possible) to accomplish a task with that specific app, learning this knowledge even before installing the app. For example, Microsoft introduced Copilot AI assistant in Office applications. By engaging with Copilot more frequently, existing users help Microsoft's GenAI learn how productivity apps can be effectively applied to a wider array of business problems. Then, with the help of the same Copilot AI (accessible via

Internet), prospective users can better assess whether Office can meet their needs. Thus, knowledge transfer through social learning is increasingly facilitated by GenAI. Moreover, GenAI models learn not only from user interactions but also from other data sources, and recent evidence suggests that LLMs can *self-train* from a limited number of examples (Hopkin 2023). Mapping into our modeling framework, as GenAI assistants become ubiquitous, continuously available, and increasingly relied upon, *fewer* first-period adopters will suffice to generate *equivalent* levels of knowledge transfer through WOM effects as compared to scenarios without GenAI. This is equivalent to the advent of GenAI being associated with a higher w (stronger, more persuasive WOM effects).

As shown in Section 6.2, when depreciation or adoption costs are present, GenAI's upward effect on w can broaden the conditions under which seeding is optimal. However, in the absence of these two effects, GenAI may not support optimality of seeding.

7.3 Impact of Gen/Agentic AI on the Estimation Prior (α)

Compared to the impact on c and w, the impact of GenAI on software users' prior (α) and individual depreciation (λ) is considerably more nuanced. We first discuss the potential influence of GenAI on α. GenAI may expand an application's usefulness by enabling deployment across a broader range of business use cases. Moreover, when the app is used in tandem with other digital assets, the limited, costly, or poor-quality repository of the latter (complementary resources) can constrain the benefit of the former. GenAI can help mitigate this issue. For instance, Adobe has integrated GenAI into Photoshop and Stock, providing users with access to a rich stream of AI-generated images and the ability to produce their own variations using AI - capabilities that enhance the creative process and complement existing resources. More broadly, GenAI can improve user productivity, enabling individuals to extract greater value from the app (Brynjolfsson et al. 2025). However, in the near term, given the novelty of GenAI, prospective users, prior to market release and without signals from other adopters, may not fully comprehend GenAI's potential to expand the product valuation (prior to taking the product for a spin for themselves). When users initially underestimate the product value ($\alpha < 1$), the *real* value they can extract may deviate even further from their prior in the presence of GenAI. Since in our model we normalize the real value (per period) to 1, a *wider* gap between real and perceived valuation translates into a *lower* α.

In the long term, firms and users alike will become increasingly familiar with GenAI's potential. Internally, prior to release, software developers could adopt practices to pre-train AI chatbots (integrated or widely available) with examples (many scientific software tools such as Matlab or Mathematica are released with documentation that already includes examples - the next step is to feed such examples to GenAI prior to release). Furthermore, prior to product release, developers may also conduct closed beta tests with select customers, during which testers can interact with widely available GenAI assistants while using the product, thereby further refining the AI's understanding of it. As a result, GenAI could become proficient at addressing product-related inquiries even before market launch. Prospective customers could then use publicly available chatbots to update their priors from the outset, improving their ability to assess the software's true value before adoption. Consequently, the introduction of GenAI can align user priors more *closely* with real valuations, which, depending on initial over- or underestimation, can mean a *lower* or a *higher* α. However, this assumes minimal hallucination, as firms risk reputational damage if pre-trained AI assistants fabricate use cases or claim non-existent features. Moreover, given the risk of data leaks, some firms may be reluctant to expose GenAI to detailed information about unreleased products, fearing premature competitive spillovers (after all, competition can accelerate development of like capabilities - ironically, also using GenAI for coding).

In sum, GenAI can bring consumer prior α either further away from or closer to the real valuation of the product.

7.4 Impact of Gen/Agentic AI on Individual Depreciation (λ)

Finally, GenAI can also influence individual depreciation. As noted earlier, GenAI can enhance productivity. On one hand users can do more with the app (hence the value of the app increases). At the same time, GenAI can also help them complete tasks *faster*. In instances in which users have a limited need for the app (e.g., a single project), completing the task faster results in less need for the app in the future. This translates to higher depreciation (*lower* λ).

On the other hand, as its name suggests, "generative" AI can repeatedly generate new content, update older content for current contexts, and even allow more users to participate in the content generation process. Consider professional simulator applications (e.g., for aviation, military combat, law enforcement, surveillance, driving, or deep-sea diving). Prior iterations of these systems operated on either pre-programmed or user-input scenarios. Once a user went through most of the pre-programmed scenarios, the value of the simulator would decline. With GenAI, simulators can generate an ongoing stream of novel scenarios, including those dynamically tailored to a trainee's progress, allowing continuous practice with minimal repetition. A parallel can be drawn to video games, where developers have long sought to enhance *replayability* by adding an element of randomization to the procedural map generation. One of the most prominent examples in this category is Blizzard's widely acclaimed Diablo game series. Every time a user restarts a campaign (potentially with a different character class), some portion of the maps/levels/non-player characters/loot would be randomly generated to create a new experience (while following some established rules for gameplay progression). Yet, such automated content rejuvenation has historically been detached from individual player preferences, and user-created map editors often proved cumbersome and underused. GenAI advances this paradigm by enabling seamless, automated content creation responsive to user prompts, performance, and progress, thereby sustaining engagement and potentially reducing depreciation (i.e., yielding a higher λ). Nevertheless, this effect might be dampened by potentially reduced variance and quality of GenAI output, a phenomenon already documented in LLMs trained on synthetic data (Shumailov et al. 2024, Bhatia 2024).

8 Conclusion

Given the rapid proliferation of software applications and the widespread adoption of the Internet, software firms now operate in increasingly congested markets where it is difficult for products to stand out. At the same time, heightened privacy concerns among consumers and regulators have curtailed firms' ability to engage in targeted marketing, and, furthermore, for many app developers, acquiring consumer data remains prohibitively expensive since most apps generate minimal revenue flow. In this environment, software producers must reconsider how to optimize consumer product discovery by evaluating how non-targeted go-to-market strategies shape multi-dimensional valuation learning on the consumer side. To this end, firms nowadays increasingly employ strategies involving some form of free consumption to stimulate valuation learning by exposing consumers to the product. In this paper, we pit two such strategies with free consumption against each other - the more traditional strategy of non-targeted seeding (which predates the emergence of digital goods) and market-wide time limited freemium (free trials available to everyone for a limited time, possible at scale only in the context of digital goods). Our central question is whether non-targeted seeding remains relevant when large-scale free trials are possible. To our knowledge, this is the first study to compare and contrast seeding with free trials (and with other non-free strategies), while jointly considering self- and social learning processes on the consumer side. Prior research shows that when time-limited freemium (TLF) is absent from the strategy choice set, seeding (S) may be optimal when consumers initially severely undervalue the product. However, our exploration reveals that this may no longer be the case when free trials are considered as well. Building on a unifying multi-period framework with learning mechanisms borrowed from established literature, we find that in a parsimonious baseline setting, in the absence of any *user adoption costs* or *individual value depreciation*, seeding is in fact never optimal once free trials are in the picture.

Intrigued by this initial finding, we set out on an additional research goal - to identify some factors that, when added to the model, would allow non-targeted seeding to re-emerge as optimal in some regions of the parameter. We find that when either of two demand characteristics - *adoption costs* or *individual depreciation* (which can be present in various combinations in industry) - are introduced, S can become the optimal strategy in scenarios in which consumers initially underestimate product value. In fact, these two factors enable each of the four business strategies to dominate in specific regions of the parameter space.

Our findings remain qualitatively consistent across a comprehensive set of robustness checks, including endogenous individual depreciation, joint modeling of depreciation and adoption costs to ensure their effects do not offset each other, more general WOM effects, imperfect self-learning, and extended time horizons capturing compounded learning effects. Additionally, we show that depreciation and adoption costs moderate the influence of WOM on the optimality of seeding. In the absence of these factors, stronger WOM effects alone cannot make seeding superior to other strategies; however, once either factor is introduced, strong WOM effects expand seeding's optimality region within the parameter space, increasing its relevance. Identifying ranges of adoption costs and individual depreciation that render seeding optimal in some regions provides actionable managerial insights. Relatedly, we map the parameter regions where each of the considered go-to-market strategies emerges as optimal, information that is also highly relevant to practitioners. Furthermore, under the main scenarios in Sections 3-5, TLF yields the highest social welfare, even though firms may prefer alternative strategies depending on parameter regions, revealing a notable misalignment between societal and firm objectives. Our study informs policy makers and regulators on when it could be socially beneficial to intervene in app markets and when not to. Solving separately the non-trivial equilibrium strategies for each of the strategies in the presence of depreciation and adoption costs adds secondary contributions - these complete characterizations can be used as building blocks to further advance the exploration of each of these strategies.

Furthermore, in Section 7, we delved into how the emergence of Gen/Agentic AI can reshape the optimal go-to-market strategy by influencing key market primitives and consumer learning dynamics. In our unconstrained analysis, seeding proved optimal only within a relatively narrow range of the parameter space. We show, however, that Gen/Agentic AI can shift the market toward, or even broaden that favorable set, in particular enhancing the viability of S. Thus, this AI revolution makes the insights of this work even more timely and consequential.

This study opens several avenues for future research. We identified two market factors that sustain seeding as a viable strategy, yet further studies may uncover additional factors and refine this theory. In the context of a longer horizon, future studies could also explore hybrid strategies that mix aspects from multiple approaches (including combining features from S and TLF), or more adaptive strategies. Moreover, the discussion in Section 7 motivates an empirical future research agenda to quantify all mentioned GenAI-induced effects on go-to-market strategies. Lastly, our analysis incorporates the role of network externalities in the social learning process. Future research can also investigate how non-targeted seeding and the other strategies fare against each other in the presence of direct network effects at the consumer utility level (whereby larger installed bases directly drive up the utility of adopters through value exchange and collaboration). Intuitively, seeding should benefit from direct network effects because the perpetual license of the seeds would ensure a minimum added boost at utility level (due to a non-zero initial installed base) for every prospective consumer at every future stage. We therefore expect our results to continue to be robust to some extent even in the presence of direct network effects. Interesting additional dynamics are likely to emerge in models that integrate both direct network and WOM effects, as non-adopters would be drawn to the market not only through WOM-based prior updating but also through expectations about the magnitude of the installed base, with the sequencing of these mechanisms potentially influencing how they perceive the benefits of such direct network effects.

References

Akbarpour, M., S. Malladi, and A. Saberi (2023). Just a few seeds more: The inflated value of network data for diffusion. Working paper. `https://web.stanford.edu/~mohamwad/NetworkSeeding.pdf`, Accessed on 12/23/2025.

Aral, S., L. Muchnik, and A. Sundararajan (2013). Engineering social contagions: Optimal network seeding in the presence of homophily. *Network Sci. 1*(2), 125–153.

Aridor, G., Y.-K. Che, B. Hollenbeck, D. McCarthy, and M. Kaiser (2025). Evaluating the impact of privacy regulation on e-commerce firms: Evidence from Apple's App Tracking Transparency. *Management Sci. forthcoming.*

Bass, M. F. (1969). A new product growth model for consumer durables. *Management Sci. 15*(5), 215–227.

Bates, J. M. and C. W. Granger (1969). The combination of forecasts. *J. Oper. Res. Society 20*(4), 451–468.

Bhatia, A. (2024). When A.I.'s output is a threat to A.I. itself. The New York Times. Aug. 25. `https://www.nytimes.com/interactive/2024/08/26/upshot/ai-synthetic-data.html`, Accessed on 12/23/2025.

Bick, A., A. Blandin, and D. J. Deming (2025). The rapid adoption of generative AI. Federal Reserve Bank of St. Louis Working Paper 2024-027, Rev. Oct. 2025, `https://doi.org/10.20955/wp.2024.027`, Accessed on 12/23/2025.

Brynjolfsson, E., D. Li, and L. Raymond (2025). Generative AI at work. *Quart. J. Econ. 140*(2), 889–942.

Chen, P., L. M. Hitt, Y. Hong, and S. Wu (2021). Measuring product type and purchase uncertainty with online product ratings: A theoretical model and empirical application. *Inform. Syst. Res. 32*(4), 1470–1489.

Chen, X., R. Van Der Lans, and T. Q. Phan (2017). Uncovering the importance of relationship characteristics in social networks: Implications for seeding strategies. *J. Marketing Res. 54*(2), 187–201.

Chen, Y.-H. and B. Jiang (2021). Dynamic pricing and price commitment of new experience goods. *Prod. & Oper. Management 30*(8), 2752–2764.

Chin, A., D. Eckles, and J. Ugander (2022). Evaluating stochastic seeding strategies in networks. *Management Sci. 68*(3), 1714–1736.

Choudhary, V. (2007). Comparison of software quality under perpetual licensing and software as a service. *J. Management Inform. Systems 24*(2), 141–165.

Cisco (2024). Cisco 2024 consumer privacy survey. `https://www.cisco.com/c/en/us/about/trust-center/consumer-privacy-survey.html`, Accessed on 12/23/2025.

Cruz, J. (2022). 91% of app revenue comes from the top 1% of publishers, but that share is shrinking. Sensor Tower Blog. Sep. `https://sensortower.com/blog/top-one-percent-downloads-1h-2022`, Accessed on 12/23/2025.

Cui, F., H.-h. Hu, W.-t. Cui, and Y. Xie (2018). Seeding strategies for new product launch: The role of negative word-of-mouth. *PloS one 13*(11), e0206736.

Cui, F., L. Wang, X. R. Luo, and X. Cui (2024). Influentials, early adopters, or random targets? Optimal seeding strategies under vertical differentiations. *Decision Support Syst. 183*, 114263.

Dou, Y., Y. J. Hu, and D. J. Wu (2017). Selling or leasing? Pricing information goods with depreciation of consumer valuation. *Inform. Syst. Res. 28*(3), 585–602.

Dou, Y., M. F. Niculescu, and D. J. Wu (2013). Engineering optimal network effects via social media features and seeding in markets for digital goods and services. *Inform. Syst. Res. 24*(1), 164–185.

Fortin, M. (2018). Windows 10 Quality approach for a complex ecosystem. Microsoft Windows Blogs, Nov. 13, https://blogs.windows.com/windowsexperience/2018/11/13/windows-10-quality-approach-for-a-complex-ecosystem/ , Accessed on 12/23/2025.

Galeotti, A. and S. Goyal (2009). Influencing the influencers: A theory of strategic diffusion. *RAND J. Econ. 40*(3), 509–532.

Ghoshal, A. (2023). Why generative AI will turbocharge low-code and no-code development. InfoWorld. Apr. 21. https://www.infoworld.com/article/3694173/why-generative-ai-will-turbocharge-low-code-and-no-code-development.html, Accessed on 12/23/2025.

Gokgoz, Z. A., M. B. Ataman, and G. H. van Bruggen (2021). There's an app for that! Understanding the drivers of mobile application downloads. *J. Business Res. 123*, 423–437.

Haenlein, M. and B. Libai (2013). Targeting revenue leaders for a new product. *J. Marketing 77*(3), 65–80.

Han, P., Y. Huo, and E. Qi (2021). Effect of seeding on the benefits of the manufacturer and retailer. *Computers & Industrial Engineering 153*, 107074.

Han, S. P., S. Park, and W. Oh (2016). Mobile app analytics: A multiple discrete-continuous choice framework. *MIS Quart. 40*(4), 983–1008.

Hautala, L. (2022). What to know about fake reviews when shopping prime day deals. CNET, Jul 12. https://www.cnet.com/tech/services-and-software/features/what-to-know-about-fake-reviews-when-shopping-prime-day-deals/, Accessed on 12/23/2025.

Hermann, E. and S. Puntoni (2024). Artificial intelligence and consumer behavior: From predictive to generative AI. *J. Business Res. 180*, 114720.

Hopkin, G. (2023). AI is already training itself, claim MIT and Google experts. AI Magazine, Feb. 9. https://aimagazine.com/articles/ai-is-already-training-itself-claim-mit-and-google-experts, Accessed on 12/23/2025.

Huang, M.-H. and R. T. Rust (2025). The GenAI future of consumer research. *J. Consumer Res. 52*(1), 4–17.

Ishihara, M. and A. T. Ching (2019). Dynamic demand for new and used durable goods without physical depreciation: The case of Japanese video games. *Marketing Sci. 38*(3), 392–416.

Jain, D., V. Mahajan, and E. Muller (1995). An approach for determining optimal product sampling for the diffusion of a new product. *J. Prod. Innovation Management 12*, 124–1356.

Jiang, Z. and S. Sarkar (2010). Speed matters: The role of free software offer in software diffusion. *J. Management Inform. Systems 26*(3), 207–239.

Kim, D. A., A. R. Hwong, D. Stafford, D. A. Hughes, A. J. O'Malley, J. H. Fowler, and N. A. Christakis (2015). Social network targeting to maximise population behaviour change: a cluster randomised controlled trial. *The Lancet 386*(9989), 145–153.

Korzynski, P., G. Mazurek, A. Altmann, J. Ejdys, R. Kazlauskaite, J. Paliszkiewicz, K. Wach, and E. Ziemba (2023). Generative artificial intelligence as a new context for management theories: analysis of ChatGPT. *Central European Management J. 31*(1), 3–13.

Lehmann, D. R. and M. Esteban-Bravo (2006). When giving some away makes sense to jump-start the diffusion process. *Marketing Letters 17*(4), 243–254.

Li, H., S. Jain, and P. Kannan (2019). Optimal design of free samples for digital products and services. *J. Marketing Res. 56*(3), 419–438.

Li, K. J. and S. Jain (2016). Behavior-based pricing: An analysis of the impact of peer-induced fairness. *Management Sci. 62*(9), 2705–2721.

Libai, B., E. Muller, and R. Peres (2005). The role of seeding in multi-market entry. *Int. J. Res. Marketing 22*(4), 375–393.

Libai, B., E. Muller, and R. Peres (2013). Decomposing the value of word-of-mouth seeding programs: Acceleration versus expansion. *J. Marketing Res. 50*(2), 161–176.

Lin, Z., Y. Zhang, and Y. Tan (2019). An empirical study of free product sampling and rating bias. *Inform. Syst. Res. 30*(1), 260–275.

Nejad, M. G. and M. Amini (2024). Designing profitable seeding programs: The effects of social network properties and consumer homophily. *J. Business Res. 173*, 114496.

Nejad, M. G., M. Amini, and E. Babakus (2015). Success factors in product seeding: The role of homophily. *J. Retailing 91*(1), 68–88.

Niculescu, M. F. and D. J. Wu (2014). Economics of free under perpetual licensing: Implications for the software industry. *Inform. Syst. Res. 25*(1), 173–199.

RevenueCat (2025). State of subscription apps 2025. Market report. `https://www.revenuecat.com/state-of-subscription-apps-2025/`, Accessed on 12/23/2025.

Roth, E. (2025). Google's Play Store lost nearly half its apps. The Verge. Apr. 30. `https://www.theverge.com/news/658932/google-play-store-lost-half-apps`, Accessed on 12/23/2025.

Russo, D. (2024). Navigating the complexity of generative AI adoption in software engineering. *ACM Transactions on Software Eng. and Methodology 33*(5), 1–50.

Schlereth, C., C. Barrot, B. Skiera, and C. Takac (2013). Optimal product-sampling strategies in social networks: How many and whom to target? *Int'l J. Electronic Commerce 18*(1), 45–72.

Shiller, B. R. (2013). Digital distribution and the prohibition of resale markets for information goods. *Quant. Marketing & Econ. 11*(4), 403–435.

Shulman, J. D., M. Cunha Jr, and J. K. Saint Clair (2015). Consumer uncertainty and purchase decision reversals: Theory and evidence. *Marketing Sci. 34*(4), 590–605.

Shumailov, I., Z. Shumaylov, Y. Zhao, N. Papernot, R. Anderson, and Y. Gal (2024). AI models collapse when trained on recursively generated data. *Nature 631*(8022), 755–759.

Statista (2025). Retention rate on day 30 of mobile Android app installs worldwide in 3rd quarter of 2024, by category. `https://www.statista.com/statistics/259329/ios-and-android-app-user-retention-rate/`, Accessed on 12/23/2025.

Tan, Y. (2024). Implications of blockchain-powered marketplace of preowned virtual goods. *Prod. & Oper. Management 33*(6), 1393–1409.

Weathers, D., S. Sharma, and S. L. Wood (2007). Effects of online communication practices on consumer perceptions of performance uncertainty for search and experience goods. *J. Retailing 83*(4), 393–401.

Wilder, B., N. Immorlica, E. Rice, and M. Tambe (2018). Maximizing influence in an unknown social network. Proc. AAAI Conference on Artificial Intelligence, 32(1).

Zhang, H., Z. Zhang, and S. Raghunathan (2021). Is more better? The divide between retailer's and manufacturers' preferences for reviews and review monetization. *MIS Quart. 45*(3), 1349–1410.

Zhang, J. and A. Seidmann (2010). Perpetual versus subscription licensing under quality uncertainty and network externality effects. *J. Management Inform. Syst. 27*(1), 39–68.

Zhang, Z., G. Nan, M. Li, and Y. Tan (2022). Competitive entry of information goods under quality uncertainty. *Management Sci. 68*(4), 2869–2888.

Electronic Companion

Don't Count Non-Targeted Seeding Out Just Yet

Y. Dou, H. Hu, M. F. Niculescu, D. J. Wu

A Summary of Key Notation for Main Setup (Sections 3-5)

Table A1: Summary of key notation.

Symbol	Description
θ	Consumer type
a	True product quality (normalized to 1 w.l.o.g.)
a_t	Consumers' perceived valuation at the beginning of period t, with $t \in \{1, 2\}$
α	Level of deviation in valuation in ex-ante (prior) beliefs
λ	Use-based value depreciation
c	Adoption costs incurred by consumers
w	Strength of WOM effects
p	Price of the product
k	Seeding ratio under S model
π	Firm's profit
N_t	Size of paying population in period t, with $t \in \{1, 2\}$
$N_{t,total}$	Size of installed base (including paying, free trial, and/or seeded consumers) in period t, with $t \in \{1, 2\}$

B Proofs of Results for the Baseline Setup

We first present the optimal strategies under each of the business models separately. The solutions for pricing and profit for *CE-PL* and *S* are reproduced from Niculescu and Wu (2014) for readers' convenience.

Proposition B.1. [*expanded from Proposition 1 in Niculescu and Wu (2014) to include social welfare*] *Under CE-PL model, the firm's optimal pricing strategy, profit, and ensuing social welfare are:*

	$0 < \alpha < 13 - 4\sqrt{10}$	$13 - 4\sqrt{10} \leq \alpha$
p^*_{CE-PL}	$\frac{2\alpha}{1-\alpha}\left(1 - \sqrt{\frac{2\alpha}{1+\alpha}}\right)$	α
π^*_{CE-PL}	$\frac{2\alpha(\sqrt{1+\alpha}-\sqrt{2\alpha})^2}{(1-\alpha)^2}$	$\frac{\alpha}{2}$
SW^*_{CE-PL}	$1 - \frac{1+2\alpha+2\alpha^2}{2(1+\alpha)(\sqrt{1+\alpha}+\sqrt{2\alpha})^2}$	$\frac{3}{4}$
Paid Adoption	in both periods	only in period 1

Proof. See Proposition 1 in Niculescu and Wu (2014) for the derivation of p^*_{CE-PL} and π^*_{CE-PL}. The social welfare derivation follows trivially. □

Proposition B.2. *Under CE-SUB model, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

	$0 < \alpha \leq 1$	$1 < \alpha \leq 3$	$\alpha > 3$
p^*_{CE-SUB}	$\tilde{p}$	$\frac{\alpha}{1+\alpha}$	$\frac{\alpha}{2}$
π^*_{CE-SUB}	$\tilde{p}\left(1-\frac{\tilde{p}}{\alpha}+1-\frac{\tilde{p}}{1+\tilde{p}-\frac{\tilde{p}}{\alpha}}\right)$	$\frac{\alpha}{1+\alpha}$	$\frac{\alpha}{4}$
SW^*_{CE-SUB}	$1-\frac{1}{2}\left(\frac{\tilde{p}}{\alpha}\right)^2-\frac{1}{2}\left(\frac{\tilde{p}}{1+\tilde{p}-\frac{\tilde{p}}{\alpha}}\right)^2$	$\frac{\alpha^2+4\alpha+1}{2(1+\alpha)^2}$	$\frac{3}{8}$
Subscription (paid adoption)	in both periods	in both periods	only in period 1

where $\tilde{p}$ *is unique solution to the equation* $2\alpha^3 - 2(\alpha-1)^2p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p = 0$ *on the interval* $(0, \alpha)$.

Proof. In period 1, consumers subscribe iff $\alpha\theta \geq p$. To make any profit, the firm is constrained to set $0 < p < \alpha$. The marginal adopter has type $\theta_1 = \frac{p}{\alpha}$ and the installed base in period 1 is $N_1 = 1 - \frac{p}{\alpha}$. All period 1 adopters learn the true quality of the product in the first period. In the beginning of period 2, the non-adopters from period 1 update their priors through social learning from $a_1 = \alpha$ to $a_2 = \alpha + (1-\alpha)\left(1-\frac{p}{\alpha}\right) = 1 + p - \frac{p}{\alpha}$. We have two cases:

- Case 1: $0 < \alpha \leq 1$.
  In this case, $a_1 \leq a_2 \leq a = 1$. All period 1 adopters will renew the subscription in period 2. The marginal customer type θ_2 satisfies $\theta_2 = \frac{p}{1+p-\frac{p}{\alpha}} \leq \theta_1$. Therefore, the number of adopters in period 2 is $N_2 = 1 - \frac{p}{1+p-\frac{p}{\alpha}}$. The firm's profit maximization problem becomes

$$\max_{0<p<\alpha} \pi_{CE-SUB} = \max_{0<p<\alpha} p\left(1 - \frac{p}{\alpha} + 1 - \frac{p}{1+p-\frac{p}{\alpha}}\right).$$

  Differentiating π_{CE-SUB} with respect to p we obtain:

$$\frac{\partial \pi_{CE-SUB}(p)}{\partial p} = \frac{2\alpha^3 - 2(\alpha-1)^2p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p}{\alpha(\alpha+(\alpha-1)p)^2}.$$

  The denominator is positive. Denote the numerator as $g(p) \triangleq 2\alpha^3 - 2(\alpha-1)^2p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p$. Thus, the sign of $\partial\pi_{CE-SUB}(p)/\partial p$ is the same as the sign of $g(p)$. Differentiating $g(p)$ w.r.t. p, we obtain:

$$\frac{\partial g(p)}{\partial p} = -2(\alpha+(\alpha-1)p)(3(\alpha-1)p-(\alpha-3)\alpha).$$

  We have two subcases:

  - If $\alpha = 1$, then $\frac{\partial g(p)}{\partial p} = -2\alpha^2(3-\alpha) < 0$ for all $p \in (0, \alpha)$.

- If $\alpha < 1$, then, $\frac{\partial g(p)}{\partial p} = 0$ has two solutions, p_1 and p_2 on the real line, but they are both outside the interval $(0, \alpha)$. More precisely, $\alpha < p_1 = \frac{(3-\alpha)\alpha}{3(1-\alpha)} < p_2 = \frac{\alpha}{1-\alpha}$. Thus, when $\alpha < 1$, $\frac{\partial g(p)}{\partial p} < 0$ for all $p \in (0, \alpha)$.

Thus, when $\alpha \in (0, 1]$, $g(p)$ is decreasing in p over $(0, \alpha)$. Given that $g(0) = 2\alpha^3 > 0 > g(\alpha) = -\alpha^4(1 + \alpha)$, there exists a unique $\tilde{p} \in (0, \alpha)$ that satisfies $g(p) = 0$. Thus, $\frac{\partial \pi_{CE-SUB}(p)}{\partial p} > 0$ when $p \in (0, \tilde{p})$ and $\frac{\partial \pi_{CE-SUB}(p)}{\partial p} < 0$ when $p \in (\tilde{p}, \alpha)$. As such $p^*_{CE-SUB} = \tilde{p}$ is the optimal price. The formulas for the optimal profit and associated social welfare follow trivially.

- Case 2: $\alpha > 1$.
  In this case, $a_1 > a_2 > a = 1$. None of the period 1 non-adopters will subscribe in period 2 (they value in period 2 the product even less than in period 1). Also, *only part* of the period 1 adopters will renew the subscription in period 2. Since all adopters from period 1 updated their priors to $a_2 = a = 1$ The marginal customer type θ_2 satisfies $\theta_2 = \min\{1, p\}$. We have two subcases:

  - Case 2-i: $0 < p < 1$.
    Then $\theta_2 = p$ and $N_2 = 1 - p$. The firm's profit maximization problem becomes:

$$\max_{0<p<1} \pi_{CE-SUB} = \max_{0<p<1} p \left(1 - \frac{p}{\alpha} + 1 - p\right).$$

    We have $\frac{\partial^2 \pi_{CE-SUB}(p)}{\partial p^2} < 0$. From FOC, we obtain the following interior solution $p^*_{CE-SUB} = \frac{\alpha}{1+\alpha}$. This leads to $\pi^*_{CE-SUB} = \frac{\alpha}{1+\alpha}$, $SW^*_{CE-SUB} = \frac{\alpha^2+4\alpha+1}{2(1+\alpha)^2}$.
  - Case 2-ii: $1 \leq p < \alpha$.
    Then $\theta_2 = 1$ and $N_2 = 0$, i.e., *none* of the period 1 adopters will renew the subscription in period 2. The firm's profit maximization is simplified to:

$$\max_{1\leq p<\alpha} \pi_{CE-SUB} = \max_{1\leq p<\alpha} p \left(1 - \frac{p}{\alpha}\right).$$

    We need to consider two subsequent subcases:

    * Case 2-ii-a: $1 < \alpha \leq 2$.
      Then we have a corner solution $p^*_{CE-SUB} = 1$, which yields $\pi^*_{CE-SUB} = \frac{\alpha-1}{\alpha}$ and $SW^*_{CE-SUB} = \frac{1}{2} - \frac{1}{2\alpha^2}$.
    * Case 2-ii-b: $\alpha > 2$.
      Then we have an interior solution $p^*_{CE-SUB} = \frac{\alpha}{2}$, which yields $\pi^*_{CE-SUB} = \frac{\alpha}{4}$ and $SW^*_{CE-SUB} = \frac{3}{8}$.

  If $1 < \alpha \leq 3$, $\frac{\alpha}{1+\alpha} \geq \max\{\frac{\alpha-1}{\alpha}, \frac{\alpha}{4}\}$. If $\alpha > 3$, then $\frac{\alpha}{4} > \frac{\alpha}{1+\alpha} > \frac{\alpha-1}{\alpha}$. Comparing π^*_{CE-SUB} values among subcases, the results follow immediately. □

Proposition B.3. *Under* TLF *model, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are* $p^*_{TLF} = \frac{1}{2}$, $\pi^*_{TLF} = \frac{1}{4}$, *and* $SW^*_{TLF} = \frac{7}{8}$, *respectively.*

Proof. Under *TLF*, all customers get the product for free in period 1, i.e., $N_{1,total} = 1$ (but the number of paying customers is $N_1 = 0$) . Consequently, in period 2, all customers update their prior on quality to $a_2 = a = 1$. Thus, customers purchase the product if and only if their types satisfy $\theta \geq p$, yielding $N_2 = 1 - p$. The firm's profit maximization problem becomes:

$$\max_{0<p<1} \pi_{TLF} = \max_{0<p<1} p(1-p),$$

which, in turn, yields $p^*_{TLF} = \frac{1}{2}$ and $\pi^*_{TLF} = \frac{1}{4}$. The social welfare is $\int_0^1 \theta d\theta = \frac{1}{2}$ for period 1 and $\int_{\frac{1}{2}}^1 \theta d\theta = \frac{3}{8}$ for period 2, which gives $SW^*_{TLF} = \frac{7}{8}$. □

Proposition B.4. [*expanded from Proposition 2 in Niculescu and Wu (2014) to include social welfare*] *Under S model, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

	$0 < \alpha < \alpha_S$	$\alpha_S \leq \alpha < 13 - 4\sqrt{10}$	$\alpha \geq 13 - 4\sqrt{10}$
k^*_S	$\frac{1-2\alpha}{2(1-\alpha)}$	0	0
p^*_S	$\frac{1}{4}$	$\frac{2\alpha}{1-\alpha}\left(1 - \sqrt{\frac{2\alpha}{1+\alpha}}\right)$	α
π^*_S	$\frac{1}{16(1-\alpha)}$	$\frac{2\alpha(\sqrt{1+\alpha}-\sqrt{2\alpha})^2}{(1-\alpha)^2}$	$\frac{\alpha}{2}$
SW^*_S	$\frac{11-16\alpha}{16(1-\alpha)}$	$1 - \frac{1+2\alpha+2\alpha^2}{2(1+\alpha)(\sqrt{1+\alpha}+\sqrt{2\alpha})^2}$	$\frac{3}{4}$
Paid adoption	only in period 2	in both periods	only in period 1

where $\alpha_S \approx 0.065$ *is the unique solution to the equation* $f_S(\alpha) = 0$ *over the interval* $(0, 13 - 4\sqrt{10})$, *with* $f_S(\alpha) \triangleq \frac{1}{16(1-\alpha)} - \frac{2\alpha}{(\sqrt{1+\alpha}+\sqrt{2\alpha})^2}$.

Proof. See Proposition 2 in Niculescu and Wu (2014) for the derivation of p^*_S and π^*_S. The social welfare derivation follows trivially. □

Lemma B.1. *If* $0 < \alpha \leq 1$, *then* $\frac{\alpha(\alpha+3)}{4(\alpha+1)} \leq \pi^*_{CE-SUB} \leq \frac{\alpha(\alpha+1)}{3\alpha+1}$.

Proof. [Derivation of the lower bound]

$$\pi^*_{CE-SUB} = \max_{0<p<\alpha} \pi_{CE-SUB}(p) \geq \pi_{CE-SUB}(p)\left|_{p=\alpha/2}\right. = \frac{\alpha(\alpha+3)}{4(\alpha+1)}.$$

[Derivation of the upper bound]
Recall from the proof of Proposition B.2 that $\tilde{p}$ satisfies $g(\tilde{p}) = 0$ and $g(p)$ is decreasing in p over $(0, \alpha)$. Given that $g(\frac{\alpha}{2}) = \frac{1}{4}(1-\alpha)\alpha^3 > 0$, we have $\frac{\alpha}{2} < \tilde{p} < \alpha$. Also, it can be easily shown that the profit function satisfies:

$$p\left(1 - \frac{p}{\alpha} + 1 - \frac{p}{-\frac{p}{\alpha} + p + 1}\right) \leq p\left(1 - \frac{p}{\alpha} + 1 - \frac{p}{\frac{\alpha}{2} - \frac{1}{2} + 1}\right), \quad \forall p \in \left(\frac{\alpha}{2}, \alpha\right).$$

Denote $h(p) \triangleq p\left(1-\frac{p}{\alpha}+1-\frac{p}{\frac{\alpha}{2}-\frac{1}{2}+1}\right)$. Then, $\pi^*_{CE-SUB} \leq h(\tilde{p})$. We next derive an upper bound for $h(\tilde{p})$. As $h(p)$ is a concave quadratic polynomial in p, we can use F.O.C to derive its maximum on $(\frac{\alpha}{2}, \alpha)$. Setting $\frac{\partial h(p)}{\partial p} = 0$, we get the interior solution $p^*_h = \frac{\alpha(\alpha+1)}{3\alpha+1} \in (\frac{\alpha}{2}, \alpha)$. Then, $\pi^*_{CE-SUB} \leq h(\tilde{p}) \leq h(p^*_h) = \frac{\alpha(\alpha+1)}{3\alpha+1}$. □

Proof of Proposition 1.
We have two cases:

- Case 1: $0 < \alpha < 1$.
  [Firm's optimal strategy] We have several subcases:
  - Case 1-i: $0 < \alpha \leq \frac{1}{2}$.
    Then, it can be easily seen that $\pi^*_{TLF} \geq \max\{\pi^*_{CE-PL}, \pi^*_S\}$. So we are left to compare $\pi^*_{TLF} = \frac{1}{4}$ with π^*_{CE-SUB}. We define

    $$\Delta(\tilde{p}(\alpha), \alpha) \triangleq \pi^*_{CE-SUB} - \pi^*_{TLF} = \tilde{p}(\alpha)\left(1-\frac{\tilde{p}(\alpha)}{\alpha}+1-\frac{\tilde{p}(\alpha)}{-\frac{\tilde{p}(\alpha)}{\alpha}+\tilde{p}(\alpha)+1}\right)-\frac{1}{4},$$

    where $\tilde{p}(\alpha)$ was defined in Prop. B.2. Form the Envelope theorem, for $\alpha \in (0, \frac{1}{2}]$, we obtain:

    $$\frac{\partial \Delta(\tilde{p}(\alpha), \alpha)}{\partial \alpha} = \tilde{p}(\alpha)^2\left(\frac{1}{\alpha^2}+\frac{\tilde{p}(\alpha)}{(\alpha+(\alpha-1)\tilde{p}(\alpha))^2}\right) > 0.$$

    Thus, $\Delta(\tilde{p}(\alpha), \alpha)$ is increasing in α for $\alpha \in (0, \frac{1}{2}]$. From Lemma B.1, we see that $\Delta(\tilde{p}(\alpha), \alpha)\Big|_{\alpha=\frac{1}{2}} = \pi^*_{CE-SUB}\,|_{\alpha=1/2} - \frac{1}{4} > \frac{\alpha(\alpha+3)}{4(\alpha+1)}\,|_{\alpha=1/2} - \frac{1}{4} = \frac{7}{24} - \frac{1}{4} > 0$. Moreover, from Lemma B.1, we have $\lim_{\alpha\downarrow 0} \Delta(\tilde{p}(\alpha), \alpha) = \lim_{\alpha\downarrow 0} \pi^*_{CE-SUB} - \frac{1}{4} \leq \lim_{\alpha\downarrow 0} \frac{\alpha(\alpha+1)}{3\alpha+1} - \frac{1}{4} = -\frac{1}{4} < 0$. Hence, there exists a unique $\bar{\alpha} \in (0, \frac{1}{2})$ such that $\Delta(\tilde{p}(\alpha), \alpha)\,|_{\alpha=\bar{\alpha}} = 0$.

    Thus, *TLF* is the dominating strategy on the interval $(0, \bar{\alpha})$, whereas *CE-SUB* is the dominant strategy on the interval $[\bar{\alpha}, \frac{1}{2}]$.

  - Case 1-ii: $\frac{1}{2} < \alpha < 1$.
    Then, it can be easily seen that $\pi^*_{CE-PL} > \pi^*_{TLF}$ and $\pi^*_{CE-PL} = \pi^*_S$ (more precisely, *S* defaults to *CE-PL*). Thus, we only have to compare π^*_{CE-PL} and π^*_{CE-SUB}. Using Lemma B.1, we have $\pi^*_{CE-SUB} \geq \frac{\alpha(\alpha+3)}{4(\alpha+1)} > \frac{\alpha}{2} = \pi^*_{CE-PL}$. Thus, *CE-SUB* is the dominant strategy.

  [Social welfare comparison] It can be shown with relative ease, through direct comparisons of closed form solutions, that $SW^*_{TLF} = \frac{7}{8} \geq \max\{SW^*_{CE-PL}, SW^*_S\}$ for all $\alpha \in (0, 1)$. Thus, we only have to compare SW^*_{TLF} with SW^*_{CE-SUB}. We have shown in the proof of Lemma B.1

that $\tilde{p}(\alpha) \in (\frac{\alpha}{2}, \alpha)$. It is straightforward to see that:

$$SW^*_{CE-SUB} = 1 - \frac{1}{2}\left(\frac{\tilde{p}}{\alpha}\right)^2 - \frac{1}{2}\left(\frac{\tilde{p}}{1+\tilde{p}-\frac{\tilde{p}}{\alpha}}\right)^2 < 1 - \frac{1}{2}\left(\frac{\tilde{p}}{\alpha}\right)^2 < \frac{7}{8} = SW^*_{TLF}.$$

Thus, *TLF* yields the highest social welfare.

- Case 2: $\alpha \geq 1$.
  It can be seen from Propositions B.1-B.4, by comparing profits and social welfare values, that *CE-PL* is the dominant strategy in terms of the profit[B-1] and *TLF* is the dominant strategy in terms of the social welfare. □

C Proofs of Results for the Setup with Individual Depreciation

We first present the optimal strategies under each of the business models separately.

Proposition C.1. *Under CE-PL model, in the presence of exogenous individual depreciation, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

	(a) $0 < \alpha < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$	(b) Otherwise
$p^*_{CE\text{-}PL}$	$\frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}$	$\frac{1}{2}\alpha(1+\lambda)$
$\pi^*_{CE\text{-}PL}$	$\frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$	$\frac{1}{4}\alpha(1+\lambda)$
$SW^*_{CE\text{-}PL}$	$\tilde{SW}_{CE\text{-}PL}$	$\frac{3(\lambda+1)}{8}$
Paid adoption	in both periods	only in period 1

where $\tilde{SW}_{\text{CE-PL}} = \frac{1}{2}\left(1+\lambda - \frac{\alpha^2(\lambda+1)^2}{\left(\alpha\lambda+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+\alpha\right)^2} - \frac{\lambda}{\left(\alpha\lambda+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+1\right)^2}\right)$.

Proof. In period 1, consumers purchase the product iff $(1+\lambda)\alpha\theta \geq p$. To make any profit, the firm is constrained to trigger adoption in period 1 (otherwise, no customer would update their priors and there will also be no adopters in period 2 either). To achieve that, the firm has to set price $p \in (0, (1+\lambda)\alpha)$. The marginal adopter has type $\theta_1 = \frac{p}{(1+\lambda)\alpha}$ and the installed base in period 1 is $N_1 = 1 - \theta_1 = 1 - \frac{p}{(1+\lambda)\alpha}$.

At the beginning of period 2, the consumers who did not adopt in period 1 update their priors via social learning from $a_1 = \alpha$ to:

$$a_2 = a_1 + N_1(1-a_1) = \alpha + (1-\alpha)\left(1 - \frac{p}{\alpha(1+\lambda)}\right) = 1 + \frac{(\alpha-1)p}{\alpha(1+\lambda)}.$$

In period 2, new consumers purchase the product if their type θ satisfies $a_2\theta \geq p$. It immediately follows the marginal potential consumer in period 2 has type $\theta_2 = \frac{p}{1+\frac{(\alpha-1)p}{\alpha(1+\lambda)}}$. We have new adopters in period 2 iff $0 \leq \theta_2 < \theta_1$. We have two cases:

[B-1] We point out that S defaults to *CE-PL* in this region as $k^* = 0$.

- Case 1: $0 < \alpha < 1$
  In this case, we have two subcases:
  - Case 1-i: $0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} < \alpha(1+\lambda)$.
    Then we have $0 < \theta_2 < \theta_1$. Then, $N_2 = \theta_1 - \theta_2 > 0$. In this case, the firm's profit maximization problem becomes:

$$\max_{0<p<\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}} \pi_{CE-PL} = \max_{0<p<\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}} p\left(1 - \frac{p}{1+\frac{(\alpha-1)p}{\alpha(\lambda+1)}}\right).$$

    It can be shown that $\frac{\partial^2 \pi_{CE-PL}}{\partial p^2} < 0$ for $p \in (0, \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha})$. Thus, it is sufficient to solve FOC:

$$\frac{\partial \pi_{CE-PL}}{\partial p} = \frac{\alpha^2(\lambda+1)^2 + (1-\alpha)p^2(\alpha\lambda+1) - 2\alpha(\lambda+1)p(\alpha\lambda+1)}{(\alpha\lambda+\alpha+(\alpha-1)p)^2} = 0.$$

    Without constrains, the FOC yields two solutions:

$$p_1 = \frac{\alpha(\lambda+1)\left(\alpha\lambda+1+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)},$$
$$p_2 = \frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}.$$

    It can be shown that $p_1 > \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$. Comparing p_2 with $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$, we get two subcases:

    * Case 1-i-a: $\alpha(\lambda+1)(\alpha\lambda+1) < 1$
      Then $0 < p_2 < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$. It immediately follows that $p^*_{CE-PL} = p_2 = \frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}$, and $\pi^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$.

    * Case 1-i-b: $\alpha(\lambda+1)(\alpha\lambda+1) \geq 1$.
      Then $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} \leq p_2$. In this case, we have the corner solution $p^*_{CE-PL} = \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$, $\pi^*_{CE-PL} = \frac{\alpha^2\lambda(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)^2}$.

  - Case 1-ii: $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} \leq p < \alpha(\lambda+1)$.
    Then $\theta_2 \geq \theta_1$. In this case, $N_2 = 0$; adoption takes place only in period 1. The firm's profit maximization problem becomes:

$$\max_{\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}\leq p<(1+\lambda)\alpha} \pi_{CE-PL} = \max_{\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}\leq p<(1+\lambda)\alpha} p\left(1 - \frac{p}{\alpha(1+\lambda)}\right).$$

    Since the function is quadratic, it is sufficient to use FOC. Unconstrained, FOC yields the following solution:

$$p_3 = \frac{1}{2}(\alpha + \alpha\lambda) < (1+\lambda)\alpha.$$

Comparing p_3 with $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$, we have two subcases:

- ∗ Case 1-ii-a: $\alpha + 2\alpha\lambda > 1$.
  Then $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} < p_3 < \alpha(\lambda+1)$, and, thus, $p^*_{CE-PL} = p_3 = \frac{1}{2}(\alpha + \alpha\lambda)$ and $\pi^*_{CE-PL} = \frac{1}{4}(\alpha + \alpha\lambda)$;

- ∗ Case 1-ii-b: $\alpha + 2\alpha\lambda \le 1$.
  Then $p_3 \le \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$. Then, we have the corner solution $p^*_{CE-PL} = \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$, $\pi^*_{CE-PL} = \frac{\alpha^2\lambda(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)^2}$.

Comparing case 1-i and case 1-ii, we can get the optimal solution and the associated social welfare for case 1:

- If $0 < \alpha < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$, then $p^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}$, $\pi^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$, and $SW^*_{CE-PL} = \frac{1}{2}\left(1+\lambda - \frac{\alpha^2(\lambda+1)^2}{\left(\alpha\lambda+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+\alpha\right)^2} - \frac{\lambda}{\left(\alpha\lambda+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+1\right)^2}\right)$;
- If $5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)} \le \alpha < 1$, then $p^*_{CE-PL} = \frac{1}{2}(\alpha+\alpha\lambda)$, $\pi^*_{CE-PL} = \frac{1}{4}(\alpha+\alpha\lambda)$, and $SW^*_{CE-PL} = \frac{3(\lambda+1)}{8}$.

- Case 2: $\alpha \ge 1$
  In this case, $a_1 > a_2 > a = 1$. None of the period 1 non-adopters will purchase in period 2. The firm's profit maximization problem is:

$$\max_{0<p<(1+\lambda)\alpha} \pi_{CE-PL} = p\left(1 - \frac{p}{\alpha(1+\lambda)}\right).$$

Since the profit is quadratic in p, we can derive the solution from FOC. We get $p^*_{CE-PL} = \frac{1}{2}\alpha(1+\lambda)$, $\pi^*_{CE-PL} = \frac{1}{4}\alpha(1+\lambda)$, and $SW^*_{CE-PL} = \frac{3(\lambda+1)}{8}$. □

Proposition C.2. *Under CE-SUB model, in the presence of exogenous individual depreciation, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

	(a) $0 < \alpha \le \lambda \le 1$	(b) $\lambda < \alpha \le \alpha^\dagger$	(c) $\alpha^\dagger < \alpha \le 1$	(d) $1 < \alpha \le \max\{3\lambda, 1\}$	(e) $\alpha > \max\{3\lambda, 1\}$
p^*_{CE-SUB}	p_a	p_b	$\frac{\alpha}{\sqrt{\alpha}+1}$	$\frac{\alpha\lambda}{\alpha+\lambda}$	$\frac{\alpha}{2}$
π^*_{CE-SUB}	$\pi_{CE-SUB,a}$	$\pi_{CE-SUB,b}$	$\frac{\alpha}{\left(\sqrt{\alpha}+1\right)^2}$	$\frac{\alpha\lambda}{\alpha+\lambda}$	$\frac{\alpha}{4}$
SW^*_{CE-SUB}	$SW_{CE-SUB,a}$	$SW_{CE-SUB,b}$	$\frac{2\sqrt{\alpha}+1}{2\left(\sqrt{\alpha}+1\right)^2}$	$\frac{1}{2}\left(1+\lambda-\frac{\lambda\left(\alpha^2+\lambda\right)}{(\alpha+\lambda)^2}\right)$	$\frac{3}{8}$
Paid adoption	in both periods	in both periods	in both periods	in both periods	only in period 1

where:

- $p_a \in \left(\frac{\alpha}{2}, \alpha\right)$ *is the unique solution to the equation* $G_{SUB,a}(p) \triangleq 2\alpha^3 - 2(\alpha-1)^2p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p = 0$ *over the interval* $(0, \alpha)$, $\pi_{CE-SUB,a} = p_a\left(2 - \frac{p_a}{\alpha} - \frac{p_a}{1+p_a-\frac{p_a}{\alpha}}\right)$, *and* $SW_{CE-SUB,a} = \frac{1}{2}\left(1 + \lambda - \frac{\lambda p_a^2}{\alpha^2} - \frac{p_a^2}{\left(1+p_a-\frac{p_a}{\alpha}\right)^2}\right)$;

- $p_b \in \left(\frac{\lambda}{2}, \lambda\right)$ *is the unique solution to the equation* $G_{SUB,b}(p) \triangleq 2\alpha^2\lambda + (\alpha-1)p^2(\alpha(\lambda-4) - 2\lambda) - 2(\alpha-1)^2p^3 + 2\alpha p(\alpha(\lambda-1) - 2\lambda) = 0$ *over the interval* $(0, \lambda)$, $\pi_{CE-SUB,b} = p_b\left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1+p_b-\frac{p_b}{\alpha}}\right)$, *and* $SW_{CE-SUB,b} = \frac{1}{2}\left(1 + \lambda - \frac{p_b^2}{\lambda} - \frac{1}{\left(1+p_b-\frac{p_b}{\alpha}\right)^2}\right)$; *and*

- *threshold* $\alpha^\dagger$ *is defined in implicit form in the proof, in equation* (C.1).

Proof. In period 1, customers subscribe iff $\alpha\theta \geq p$. To make any profit, the firm is constrained to set $0 < p < \alpha$. The marginal adopter has type $\theta_1 = \frac{p}{\alpha}$ and the installed base in period 1 is $N_1 = 1 - \frac{p}{\alpha}$. All period 1 adopters learn the true quality of the product in the first period. At the beginning of period 2, the period 1 non-adopters update their priors via social learning from $a_1 = \alpha$ to $a_2 = \alpha + (1-\alpha)\left(1 - \frac{p}{\alpha}\right) = 1 + p - \frac{p}{\alpha}$. We have two cases:

- Case 1: $0 < \alpha \leq 1$.
  In this case, $a_1 \leq a_2 \leq a = 1$. The marginal customer type for period 1 non-adopters at the beginning of period 2 is $\theta_2 = \frac{p}{1+p-\frac{p}{\alpha}} < \theta_1$. Thus, all customers with types $\theta \in [\theta_2, \theta_1)$ are new adopters in period 2 (i.e., fresh subscribers). For period 1 adopters, while their valuation of the product increased, due to individual depreciation, there is a limited residual value that they can extract in period 2. These past adopters make another decision at the beginning of period 2 on whether to renew subscription or abandon the product. A period 1 adopter with type θ will renew subscription in period 2 iff $p \leq \lambda\theta$. We get several subcases:

  – Case 1-i: $\alpha \leq \lambda \leq 1$.
    Then $p/\lambda \leq \theta_1$. All period 1 subscribers renew the subscription in period 2. The profit maximization becomes:

$$\max_{0<p<\alpha} \pi_{CE-SUB} = \max_{0<p<\alpha} p\left(1 - \frac{p}{\alpha} + 1 - \frac{p}{1+p-\frac{p}{\alpha}}\right).$$

    It can be shown that SOC is satisfied ($\frac{\partial^2 \pi_{CE-SUB}}{\partial p^2} < 0$). Hence, FOC is sufficient to determine the optimal price:

$$\frac{\partial \pi_{CE-SUB}}{\partial p} = \frac{2\alpha^3 - 2(\alpha-1)^2p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p}{\alpha(\alpha + (\alpha-1)p)^2}.$$

    When solving FOC ($\frac{\partial \pi_{CE-SUB}}{\partial p} = 0$), it is enough to look at the numerator.
    Denote $G_{SUB,a}(p) \triangleq 2\alpha^3 - 2(\alpha-1)^2p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p = 0$. It can be easily shown that $G_{SUB,a}(p)$ is decreasing in $\left(-\infty, \frac{(3-\alpha)\alpha}{3(1-\alpha)}\right)$, increasing in $\left(\frac{(3-\alpha)\alpha}{3(1-\alpha)}, \frac{\alpha}{1-\alpha}\right)$, and decreasing in $\left(\frac{\alpha}{1-\alpha}, +\infty\right)$. Moreover, $\alpha < \frac{(3-\alpha)\alpha}{3(1-\alpha)} < \frac{\alpha}{1-\alpha}$.

Evaluating $G_{SUB,a}(p)$ at various threshold points allows us to further narrow the bounds for p_a:

$$G_{SUB,a}(0) > G_{SUB,b}\left(\frac{\alpha}{2}\right) > 0 > G_{SUB,a}(\alpha) > G_{SUB,a}\left(\frac{(3-\alpha)\alpha}{3(1-\alpha)}\right),$$
$$G_{SUB,a}\left(\frac{\alpha}{1-\alpha}\right) < 0.$$

Thus, $G_{SUB,a}(p) = 0$ has a unique solution $p_a \in \left(\frac{\alpha}{2}, \alpha\right)$ over the real line, which is also the price value maximizing the profit in this region. More precisely, $\frac{\partial \pi_{CE-SUB}}{\partial p} > 0$ for $p \in (0, p_a)$ and $\frac{\partial \pi_{CE-SUB}}{\partial p} < 0$ for $p \in (p_a, \alpha)$. The formulas for the optimal profit and associated social welfare follow trivially.

– Case 1-ii: $\lambda < \alpha \leq 1$.
We explore two subcases:

  * Case 1-ii-a: $\lambda < p < \alpha$.
  Then $p/\lambda > 1 > \theta_1$. In this case, all period 1 subscribers (customers with type $\theta \in [\theta_1, 1]$) unsubscribe in period 2. The profit maximization problem becomes:

$$\max_{\lambda<p<\alpha} \pi_{CE-SUB} = \max_{\lambda<p<\alpha} p\left(1 - \frac{p}{\alpha} + \frac{p}{\alpha} - \frac{p}{1+p-\frac{p}{\alpha}}\right) = \max_{\lambda<p<\alpha} p\left(1 - \frac{p}{1+p-\frac{p}{\alpha}}\right).$$

  It can be shown that $\frac{\partial^2 \pi_{CE-SUB}}{\partial p^2} < 0$. Thus, FOC is sufficient to determine the optimal price. Solving the unconstrained FOC:

$$\frac{\partial \pi_{CE-SUB}}{\partial p} = \frac{\alpha^2 + p^2 - \alpha(p+2)p}{(\alpha + (\alpha-1)p)^2} = 0,$$

  we get two candidate solutions:

$$p_1 = \frac{\alpha}{1-\sqrt{\alpha}} \text{ and } p_2 = \frac{\alpha}{1+\sqrt{\alpha}}.$$

  It immediately follows that $p_1 > \alpha$ and $p_2 < \alpha$. Thus, p_2 is the only feasible candidate against the upper bound. Comparing p_2 and λ (the lower bound), we get two subcases:

    · Case 1-ii-a1: $\frac{\alpha}{1+\sqrt{\alpha}} \leq \lambda$.
    Then $p^*_{CE-SUB} \downarrow \lambda$, which is a corner solution and is weakly dominated by the case when $p \leq \lambda$ (case 1-ii-b).

    · Case 1-ii-a2: $\frac{\alpha}{1+\sqrt{\alpha}} > \lambda$.
    We point out that this subcase is feasible only when $0 < \lambda < \frac{1}{2}$ and $\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2} < \alpha \leq 1$. Then $p^*_{CE-SUB} = \frac{\alpha}{\sqrt{\alpha}+1}$, $\pi^*_{CE-SUB} = \frac{\alpha}{(\sqrt{\alpha}+1)^2}$, and $SW^*_{CE-SUB} =$

$\frac{2\sqrt{\alpha}+1}{2(\sqrt{\alpha}+1)^2}$.

* Case 1-ii-b: $p \le \lambda$.
Then $1 \ge p/\lambda > \theta_1$. In this case, period 1 subscribers with type $\theta \in [\theta_1, p/\lambda)$ unsubscribe in period 2. The profit maximization problem becomes:

$$\max_{p\le\lambda} \pi_{CE-SUB} = \max_{p\le\lambda} p\left(1-\frac{p}{\alpha}+1-\frac{p}{\lambda}+\frac{p}{\alpha}-\frac{p}{1+p-\frac{p}{\alpha}}\right) = \max_{p\le\lambda} p\left(2-\frac{p}{\lambda}-\frac{p}{1+p-\frac{p}{\alpha}}\right).$$

It can be shown that $\frac{\partial^2\pi_{CE-SUB}}{\partial p^2} < 0$. Thus, FOC is sufficient to determine the optimal price. The FOC of the profit function is:

$$\frac{\partial \pi_{CE-SUB}}{\partial p} = \frac{2\alpha^2\lambda + (\alpha-1)p^2(\alpha(\lambda-4)-2\lambda) - 2(\alpha-1)^2p^3 + 2\alpha p(\alpha(\lambda-1)-2\lambda)}{\lambda(\alpha+\alpha p-p)^2} = 0.$$

Denote $G_{SUB,b}(p) \triangleq 2\alpha^2\lambda+(\alpha-1)p^2(\alpha(\lambda-4)-2\lambda)-2(\alpha-1)^2p^3+2\alpha p(\alpha(\lambda-1)-2\lambda)$. It can be easily shown that $G_{SUB,b}(p)$ is decreasing in $\left(-\infty, \frac{\alpha(1-\lambda)+2\lambda}{3(1-\alpha)}\right)$, increasing in $\left(\frac{\alpha(1-\lambda)+2\lambda}{3(1-\alpha)}, \frac{\alpha}{1-\alpha}\right)$, and decreasing in $\left(\frac{\alpha}{1-\alpha}, +\infty\right)$. Moreover, under the current case, $\lambda < \frac{\alpha(1-\lambda)+2\lambda}{3(1-\alpha)} < \frac{\alpha}{1-\alpha}$.
Evaluating $G_{SUB,b}(p)$ at various threshold points allows us to further narrow the bounds for p_b. In particular, since we are in the case $\lambda < \alpha \le 1$, we have:

$$G_{SUB,b}(0) > G_{SUB,b}\left(\frac{\lambda}{2}\right) > 0 > G_{SUB,b}(\lambda) > G_{SUB,b}\left(\frac{\alpha(1-\lambda)+2\lambda}{3(1-\alpha)}\right),$$
$$G_{SUB,b}\left(\frac{\alpha}{1-\alpha}\right) < 0.$$

Thus, $G_{SUB,b}(p) = 0$ has a unique solution $p_b \in \left(\frac{\lambda}{2}, \lambda\right)$ over the real line, which is also the price value maximizing the profit in this region. More precisely, $\frac{\partial\pi_{CE-SUB}}{\partial p} > 0$ for $p \in (0, p_b)$ and $\frac{\partial\pi_{CE-SUB}}{\partial p} < 0$ for $p \in (p_b, \lambda)$. The formulas for the optimal profit and associated social welfare follow trivially.

We next need to compare the optimal profits under cases 1-ii-a2 and 1-ii-b for the region in which we can simultaneously have $\lambda < \alpha \le 1$ and $\frac{\alpha}{1+\sqrt{\alpha}} > \lambda$. As mentioned above (under the discussion of case 1-ii-a2), that region is characterized by $0 < \lambda < \frac{1}{2}$ and $\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2} < \alpha \le 1$. Define the difference between the optimal profits under cases 1-ii-b and 1-ii-a2 as:

$$\Xi(p_b(\alpha,\lambda),\alpha,\lambda) \triangleq p_b(\alpha,\lambda)\left(2-\frac{p_b(\alpha,\lambda)}{\lambda}-\frac{p_b(\alpha,\lambda)}{1+p_b(\alpha,\lambda)-\frac{p_b(\alpha,\lambda)}{\alpha}}\right)-\frac{\alpha}{(\sqrt{\alpha}+1)^2},$$

where $p_b(\alpha,\lambda)$ is the unique solution mentioned in case 1-ii-b. From the Envelope theorem, and using $p_b < \lambda < \frac{\alpha}{1+\sqrt{\alpha}}$, we obtain:

$$\begin{aligned}\frac{\partial \Xi(p_b(\alpha,\lambda),\alpha,\lambda)}{\partial \alpha} &= \frac{p_b(\alpha,\lambda)^3}{(\alpha-(1-\alpha)p_b(\alpha,\lambda))^2} - \frac{1}{(\sqrt{\alpha}+1)^3} \\ &= \frac{p_b(\alpha,\lambda)^3(1+\sqrt{\alpha})^3 - (\alpha-(1-\alpha)p_b(\alpha,\lambda))^2}{(\alpha-(1-\alpha)p_b(\alpha,\lambda))^2(1+\sqrt{\alpha})^3} \\ &< \frac{\alpha^3 - (\alpha-(1-\alpha)\frac{\alpha}{1+\sqrt{\alpha}})^2}{(\alpha-(1-\alpha)p_b(\alpha,\lambda))^2(1+\sqrt{\alpha})^3} = 0.\end{aligned}$$

Thus, $\Xi(p_b(\alpha,\lambda),\alpha,\lambda)$ is decreasing in α.

Note that, since $p_b(\cdot)$ maximizes the profit under case 1-ii-b, it also maximizes $\Xi(p,\cdot)$ under the feasible region and it is a strictly interior solution. As such, since $\Xi(p_b(\alpha,\lambda),\alpha,\lambda) > \Xi(p,\alpha,\lambda)\,|_{p=\lambda}$ for all $\alpha \in \left(\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}, 1\right]$ when $\lambda < \frac{1}{2}$. By applying this inequality, the fact that $\frac{\alpha}{(1+\sqrt{\alpha})^2} = \frac{\lambda^2}{\alpha}$ when $\alpha = \frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}$, and a few algebraic manipulations of the grouped expressions, we get the sign of Ξ at that lower boundary for α:

$$\begin{aligned}\Xi(p_b(\alpha,\lambda),\alpha,\lambda)\Big|_{\alpha=\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}),0<\lambda<\frac{1}{2}} &> \Xi(p,\alpha,\lambda)\Big|_{p=\lambda,\alpha=\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2},0<\lambda<\frac{1}{2}} \\ &= \lambda\left(2-\frac{\lambda}{\lambda}-\frac{\lambda}{1+\lambda-\frac{\lambda}{\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}}}\right) - \frac{2\lambda}{\lambda+2+\sqrt{\lambda^2+4\lambda}} \\ &= 0.\end{aligned}$$

At the upper boundary, when $\alpha = 1$, we can directly solve $p_b(1,\lambda) = \frac{\lambda}{1+\lambda}$. Thus, $\Xi(p_b(1,\lambda),1,\lambda) = \frac{\lambda}{1+\lambda} - \frac{1}{4}$. It immediately follows that:

$$\Xi(p_b(1,\lambda),1,\lambda)\begin{cases} <0 & \text{if } 0<\lambda<\frac{1}{3}, \\ \geq 0 & \text{if } \frac{1}{3}\leq\lambda<\frac{1}{2}. \end{cases}$$

Given that $\Xi(p_b(\alpha,\lambda),\alpha,\lambda)$ is decreasing in α, then, $\Xi(p_b(\alpha,\lambda),\alpha,\lambda) \geq 0$ when $\frac{1}{3} \leq \lambda < \frac{1}{2}$ and $\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2} < \alpha \leq 1$. As such, in this region, profit under case 1-ii-b dominates profit under case 1-ii-a2.

However, when $0 < \lambda < \frac{1}{3}$, we have a single crossing. In other words, there exits a unique threshold $\tilde{\alpha}^\dagger \in \left(\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}, 1\right]$ such that $\Xi(p_b(\tilde{\alpha}^\dagger,\lambda),\tilde{\alpha}^\dagger,\lambda) = 0$, $\Xi(p_b(\alpha,\lambda),\alpha,\lambda) > 0$ for all $\alpha \in (\lambda,\tilde{\alpha}^\dagger)$, and $\Xi(p_b(\alpha,\lambda),\alpha,\lambda) < 0$ for all $\alpha \in (\tilde{\alpha}^\dagger, 1]$. Denote $\alpha^\dagger$ as:

$$\alpha^\dagger \triangleq \begin{cases} \tilde{\alpha}^\dagger, & \text{if } 0<\lambda<\frac{1}{3}, \\ 1, & \text{if } \frac{1}{3}\leq\lambda\leq 1. \end{cases} \tag{C.1}$$

Then, we obtain that Case 1-ii-b dominates Case 1-ii-a when $\lambda < \alpha < \alpha\dagger$ and Case 1-ii-a dominates Case 1-ii-b when $\alpha^\dagger \leq \alpha \leq 1$. Defining $\alpha^\dagger$ as in (C.1) ensures that region

$\alpha^\dagger \leq \alpha \leq 1$ vanishes if feasibility conditions are not met.

- Case 2: $\alpha > 1$.
  In this case, $a_1 > a_2 > a = 1$. None of period 1 non-adopters will subscribe in period 2. Also, *only part* of the period 1 adopters will renew the subscription in period 2 because of tandem pressure from both the downward updating of the valuation and the individual depreciation. The marginal subscriber in period 2 has type $\theta_2 = \min\left\{1, \frac{p}{\lambda}\right\} > \theta_1$. We have two subcases:

  - Case 2-i: $0 < p < \lambda$.
    Then, we have $\theta_2 = \frac{p}{\lambda}$ and $N_2 = 1 - \frac{p}{\lambda}$. The firm's profit maximization problem becomes:

    $$\max_{0<p<\lambda} \pi_{CE-SUB} = \max_{0<p<\lambda} p\left(1 - \frac{p}{\alpha} + 1 - \frac{p}{\lambda}\right).$$

    We have $\frac{\partial^2 \pi_{CE-SUB}}{\partial p^2} < 0$. From FOC, we obtain the following interior solution $p^*_{CE-SUB} = \frac{\alpha\lambda}{\alpha+\lambda}$, $\pi^*_{CE-SUB} = \frac{\alpha\lambda}{\alpha+\lambda}$.

  - Case 2-ii: $\lambda \leq p < \alpha$.
    Then, we have $\theta_2 = 1$ and $N_2 = 0$. The firm's profit maximization problem becomes:

    $$\max_{\lambda\leq p<\alpha} \pi_{CE-SUB} = \max_{\lambda\leq p<\alpha} p\left(1 - \frac{p}{\alpha}\right).$$

    We have two subcases:

    * Case 2-ii-a: $\alpha \leq 2\lambda$.
      This case is feasible only if $\frac{1}{2} < \lambda < 1$. Then, $p^*_{CE-SUB} = \lambda$ and $\pi^*_{CE-SUB} = \lambda\left(1 - \frac{\lambda}{\alpha}\right)$. However, we do notice that $\left(1 - \frac{\lambda}{\alpha}\right) < \frac{\alpha\lambda}{\alpha+\lambda}$. As such, case 2-i dominates case 2-ii-a and we do not have to consider case 2-ii-a going further.
    * Case 2-ii-b: $\alpha > \max\{2\lambda, 1\}$.
      Then, $p^*_{CE-SUB} = \frac{\alpha}{2}$ and $\pi^*_{CE-SUB} = \frac{\alpha}{4}$.

    Comparing profits under cases 2-i and 2-ii-b, we get:

    * If $\alpha \leq 3\lambda$, then $p^*_{CE-SUB} = \frac{\alpha\lambda}{\alpha+\lambda}$, $\pi^*_{CE-SUB} = \frac{\alpha\lambda}{\alpha+\lambda}$, and $SW^*_{CE-SUB} = \frac{1}{2}\Big(1 + \lambda - \frac{\lambda(\alpha^2+\lambda)}{(\alpha+\lambda)^2}\Big)$. We point out that this case is only feasible when $\lambda > \frac{1}{3}$. This is why we define this region as $1 < \alpha \leq \max\{1, 3\lambda\}$ in the text of the proposition and we point out this region vanishes when $\lambda < \frac{1}{3}$.
    * If $\alpha > \max\{1, 3\lambda\}$, $p^*_{CE-SUB} = \frac{\alpha}{2}$, $\pi^*_{CE-SUB} = \frac{\alpha}{4}$, and $SW^*_{CE-SUB} = \frac{3}{8}$. □

Proposition C.3. *Under TLF model, in the presence of exogenous individual depreciation, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are given by* $p^*_{TLF} = \frac{\lambda}{2}$, $\pi^*_{TLF} = \frac{\lambda}{4}$, *and* $SW^*_{TLF} = \frac{3\lambda}{8} + \frac{1}{2}$.

Proof. Under *TLF*, all consumers get the product for free in period 1, i.e., $N_{1,total} = 1$ (but the number of paying customers is $N_1 = 0$). Consequently, in period 2, all customers update their prior on quality to $a_2 = a = 1$. Taking into account depreciation, customers purchase the product in period 2 iff their types satisfy $\theta\lambda \geq p$. The firm's profit maximization problem is:

$$\max_{0<p<\lambda} \pi = p\left(1 - \frac{p}{\lambda}\right),$$

which yields $p^*_{TLF} = \frac{\lambda}{2}$, $\pi^*_{TLF} = \frac{\lambda}{4}$, and $SW^*_{TLF} = \frac{3\lambda}{8} + \frac{1}{2}$. □

Proposition C.4. *Under S model, in the presence of exogenous individual depreciation, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

	(a) $0 < \alpha < \alpha^\ddagger$	(b) $\alpha^\ddagger \leq \alpha < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$	(c) $\alpha \geq 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$
k^*_S	$\frac{1-2\alpha}{2(1-\alpha)}$	0	0
p^*_S	$\frac{1}{4}$	$\frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}$	$\frac{1}{2}\alpha(1+\lambda)$
π^*_S	$\frac{1}{16(1-\alpha)}$	$\frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$	$\frac{1}{4}\alpha(1+\lambda)$
SW^*_S	$\frac{4\lambda+7-8\alpha(\lambda+1)}{16(1-\alpha)}$	$\tilde{SW}_{CE-PL}$	$\frac{3(1+\lambda)}{8}$
Paid adoption	only in period 2	in both periods	only in period 1

where threshold $\alpha^\ddagger$ *is the unique solution to equation* $\alpha(32\alpha(\lambda+1)(8\alpha(\lambda+1)-6\lambda-7)+32\lambda+33)-1 = 0$ *over the interval* $\left(\frac{2\lambda(6\lambda+13)+14-(\lambda+1)\sqrt{48\lambda(3\lambda+5)+97}}{48(\lambda+1)^2}, \frac{1}{4(\lambda+1)}\right)$.

Proof. First, we point out that *CE-PL* is a particular case of *S* with seeding ratio set to zero. Throughout the proof, we will show that in certain regions *CE-PL* dominates *S* with non-zero seeding ratio - that is equivalent to saying that the optimal seeding ratio will be 0 in those regions (i.e., *S* defaults to *CE-PL*).

If $\alpha \geq 1$, seeding brings no benefit as any social learning calibrates perceived valuations downwards, and, as such, *S* defaults to *CE-PL*.

Thus, we are left to explore the non-trivial case of $0 < \alpha < 1$. We have two cases:

- Case 1: $0 < p < (1+\lambda)\alpha$.
  There are paying adopters in period 1 (potentially alongside seeded customers if $k > 0$). The marginal paying customer in period 1 has type $\theta_1 = \frac{p}{\alpha(1+\lambda)}$. Then, the total number of adopters in period 1 is $N_{1,total} = (1-k)\left(1 - \frac{p}{\alpha(1+\lambda)}\right) + k$. In period 2, the potential customers who have not adopted in period 1 update their prior beliefs via social learning as follows:

  $$a_2 = a_1 + N_{1,total}(1-a_1) = \alpha + (1-\alpha)\left((1-k)\left(1 - \frac{p}{\alpha(1+\lambda)}\right) + k\right) = 1 - \frac{(1-\alpha)(1-k)p}{\alpha(1+\lambda)}.$$

  A customer of type θ who has not adopted in period 1 (via paying for license or through the seeding program) will adopt in period 2 iff $\theta_1 > \theta \geq \theta_2 = \frac{p}{1-\frac{(1-\alpha)(1-k)p}{\alpha(1+\lambda)}}$. Comparing θ_1 and θ_2, we have:

  $$\theta_1 > \theta_2 \iff 0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}.$$

We have two cases:

- Case 1-i: $\alpha + \alpha\lambda \geq 1$.
  In this case, we have $\theta_2 > \theta_1$ for any $k \in [0, 1)$. There are no paying adopters in period 2. The firm's profit maximization problem becomes:

$$\max_{0<p<(1+\lambda)\alpha,\, 0\leq k<1} \pi_S = \max_{0<p<(1+\lambda)\alpha,\, 0\leq k<1} p(1-k)\left(1 - \frac{p}{\alpha(1+\lambda)}\right).$$

  It trivially follows that $k_S^* = 0$. S defaults to *CE-PL*.

- Case 1-ii: $\alpha + \alpha\lambda < 1$.
  We have:

$$\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} < (1+\lambda)\alpha \quad \Longleftrightarrow \quad 0 \leq k < \frac{\alpha\lambda}{1-\alpha}.$$

  Subsequently, we have several subcases:

  * Case 1-ii-a: $0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$ and $0 \leq k < \frac{\alpha\lambda}{1-\alpha}$.
    In this case, $\theta_2 \leq \theta_1$. Customers with type $\theta \in [\theta_2, \theta_1)$, who have not been seeded in period 1, adopt in period 2. The firm's profit maximization problem becomes:

$$\max_{0<p<\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)},\, 0\leq k<\frac{\alpha\lambda}{1-\alpha}} \pi_S = \max_{0<p<\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)},\, 0\leq k<\frac{\alpha\lambda}{1-\alpha}} p(1-k)\left(1 - \frac{p}{1-\frac{(1-\alpha)(1-k)p}{\alpha(1+\lambda)}}\right).$$

    Taking first order derivative of the profit w.r.t. p, we get:

$$\frac{\partial \pi_S}{\partial p} = \frac{(1-k)\left[\alpha^2(\lambda+1)^2 + (1-\alpha)(1-k)p^2(\alpha\lambda + (\alpha-1)k + 1) - 2\alpha(\lambda+1)p(\alpha\lambda - (1-\alpha)k + 1)\right]}{((1-\alpha)(1-k)p - \alpha(\lambda+1))^2}.$$

    The denominator is always positive. We define the numerator as a function:

$$\eta(p) \triangleq \alpha^2(\lambda+1)^2 + (1-\alpha)(1-k)p^2(\alpha\lambda - (1-\alpha)k + 1) - 2\alpha(\lambda+1)p(\alpha\lambda - (1-\alpha)k + 1).$$

    $\eta(p)$ is convex in p. Solving in unconstrained form the equation $\eta(p) = 0$, we obtain two candidate solutions:

$$p_1 = \frac{\alpha(\lambda+1)(\alpha\lambda - (1-\alpha)k + 1) - \sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda - (1-\alpha)k + 1)}}{(1-\alpha)(1-k)(\alpha\lambda - (1-\alpha)k + 1)},$$
$$p_2 = \frac{\alpha(\lambda+1)(\alpha\lambda - (1-\alpha)k + 1) + \sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda - (1-\alpha)k + 1)}}{(1-\alpha)(1-k)(\alpha\lambda - (1-\alpha)k + 1)}.$$

    It can be easily shown that $p_1 > 0$ and $p_2 > \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. Moreover:

$$p_1 < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} \quad \Longleftrightarrow \quad \frac{\alpha(1+\lambda)(1+\alpha\lambda) - 1}{\alpha(1-\alpha)(1+\lambda)} < k.$$

    We need to consider several subcases:

    · Case 1-ii-a-I: $\alpha(1+\lambda)(1+\alpha\lambda) \geq 1$.

□ Case 1-ii-a-I1: $0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$ and $0 \le k \le \frac{\alpha(1+\lambda)(1+\alpha\lambda)-1}{\alpha(1-\alpha)(1+\lambda)}$.
In this case, $p_1 \ge \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. As such, $\eta(p) > 0$ for all $0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. Thus, $\pi_S(p)$ is strictly increasing in p in this region and the profit in this case is strictly dominated by the profit under Case 1-ii-b.

□ Case 1-ii-a-I2: $0 \le p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$ and $\frac{\alpha(1+\lambda)(1+\alpha\lambda)-1}{\alpha(1-\alpha)(1+\lambda)} < k < \frac{\alpha\lambda}{1-\alpha}$.
In this case, $p_1 < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. As such, $\eta(p) > 0$ for all $p \in (0, p_1)$ and $\eta(p) < 0$ for all $p \in \left(p_1, \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}\right)$. Thus, $p_S^* = p_1$. The profit function can be simplified to:

$$\pi_S = \frac{-2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda-(1-\alpha)k+1)} + \alpha(\lambda+1)(2\alpha\lambda+\alpha-(1-\alpha)k+1)}{(1-\alpha)^2(1-k)}.$$

It is straightforward to show that $\alpha > 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$ in this case, which corresponds to the second case under *CE-PL*. For any $k \in \left(\frac{\alpha(1+\lambda)(1+\alpha\lambda)-1}{\alpha(1-\alpha)(1+\lambda)}, \frac{\alpha\lambda}{1-\alpha}\right)$, it can be easily shown that $\pi_S(k) < \frac{1}{4}\alpha(1+\lambda) = \pi^*_{CE-PL}$. Therefore, this case is sub-optimal, as it is dominated by not seeding anyone.

· Case 1-ii-a-II: $\alpha(1+\lambda)(1+\alpha\lambda) < 1$.
In this case, it immediately follows that $\frac{\alpha(1+\lambda)(1+\alpha\lambda)-1}{\alpha(1-\alpha)(1+\lambda)} < 0 \le k$. Thus, $p_1 < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. Similar to case 1-ii-a-I2, we have $p_S^* = p_1$. Following the same steps in Case 1-ii-a, we get $k_S^* = 0$. S defaults to *CE-PL*.

* Case 1-ii-b: $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} \le p < (1+\lambda)\alpha$ and $0 \le k < \frac{\alpha\lambda}{1-\alpha}$.
In this case, $\theta_2 \ge \theta_1$. There are no new adopters in period 2. The firm's profit maximization problem becomes:

$$\max_{\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} \le p < (1+\lambda)\alpha,\, 0 \le k < \frac{\alpha\lambda}{1-\alpha}} \pi_S = \max_{\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} \le p < (1+\lambda)\alpha,\, 0 \le k < \frac{\alpha\lambda}{1-\alpha}} p(1-k)\left(1 - \frac{p}{\alpha(1+\lambda)}\right).$$

It trivially follows that $k_S^* = 0$. S defaults to *CE-PL*.

* Case 1-ii-c: $0 < p < (1+\lambda)\alpha$ and $\frac{\alpha\lambda}{1-\alpha} \le k < 1$.
In this case, $p < (1+\lambda)\alpha \le \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. Then, $\theta_2 \le \theta_1$. Customers with type $\theta \in [\theta_2, \theta_1)$, who have not been seeded in period 1, adopt in period 2. The firm's profit maximization problem becomes:

$$\max_{0<p<(1+\lambda)\alpha,\, \frac{\alpha\lambda}{1-\alpha} \le k < 1} \pi_S = \max_{0<p<(1+\lambda)\alpha,\, \frac{\alpha\lambda}{1-\alpha} \le k < 1} p(1-k)\left(1 - \frac{p}{1 - \frac{(1-\alpha)(1-k)p}{\alpha(1+\lambda)}}\right).$$

Following the same steps as in Case 1-ii-a, we get the same two solutions, p_1 and p_2, to the equation $\eta(p) = 0$. It can be easily shown that $p_1 > 0$ and $p_2 > (1+\lambda)\alpha$.

Moreover:

$$p_1 < (1+\lambda)\alpha \quad \Longleftrightarrow \quad \frac{\alpha\lambda}{1-\alpha} \leq k < \min\left\{\frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)}, 1\right\}.$$

We have several subcases:

· Case 1-ii-c-I: $\alpha\lambda + \sqrt{\alpha(\lambda+1)(\alpha\lambda+\alpha+4)} + \alpha < 2$.

Then, it can be shown that $\frac{\alpha\lambda}{1-\alpha} < \frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)} < 1$.

□ Case 1-ii-c-I1: $\frac{\alpha\lambda}{1-\alpha} \leq k < \frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)}$.
In this case we have $p_1 < (1+\lambda)\alpha$. Then, we have the interior solution $p_S^* = p_1$. The profit is simplified to:

$$\pi_S = \frac{\alpha(\lambda+1)\left(-2\sqrt{\alpha(\lambda+1)(\alpha\lambda-(1-\alpha)k+1)} + (2\alpha\lambda+\alpha-(1-\alpha)k+1)\right)}{(1-\alpha)^2(1-k)}.$$

It is straightforward to show that $0 < \alpha < 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$ in this case, which corresponds to the first case under *CE-PL*. The first order derivative w.r.t. k satisfies $\frac{\partial\pi_S}{\partial k} < 0$. Hence, we have corner solution $k_S^* = \frac{\alpha\lambda}{1-\alpha}$. The optimal profit is simplified to:

$$\begin{aligned}\pi_S^* &= \frac{\alpha(\lambda+1)\left(\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)(1-\alpha-\alpha\lambda)} \\ &< \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2} = \pi_{CE-PL}^*.\end{aligned}$$

Therefore, this case is sub-optimal.

□ Case 1-ii-c-I2: $\frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)} \leq k < 1$.
In this case, we have $p_1 \geq (1+\lambda)\alpha$. We can see that, for any k in this region, $\pi_S(p)$ is strictly increasing in p and the profit in this case is strictly dominated by the profit under Case 2.

* Case 1-ii-c-II: $\alpha\lambda + \sqrt{\alpha(\lambda+1)(\alpha\lambda+\alpha+4)} + \alpha \geq 2$.
Then, it can be shown that $\frac{\alpha\lambda}{1-\alpha} < 1 \leq \frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)}$. As such, when $\frac{\alpha\lambda}{1-\alpha} \leq k \leq 1$, we have $p_1 < (1+\lambda)\alpha$, and, thus, we have the interior solution $p_S^* = p_1$. Following the same step in Case 1ii-c-I1, we get $k_S^* = \frac{\alpha\lambda}{1-\alpha}$ and, following the same reasoning, it can be shown that this case is sub-optimal as well.

In summary, we have shown that Case 1 either defaults to *CE-PL* or is strictly dominated by *CE-PL*.

- Case 2: $p \geq (1+\lambda)\alpha$.
  In this case, there are only seeded consumers in period 1 (i.e., no unseeded customer is willing to pay for the product based on priors). Hence, $N_{1,total} = k$. At the beginning of period 2, the un-seeded customers update their priors to $a_2 = \alpha + (1-\alpha)k$. The firm's profit maximization problem becomes:

$$\max_{p \geq (1+\lambda)\alpha,\, 0 \leq k < 1} \pi_S = \max_{p \geq (1+\lambda)\alpha,\, 0 \leq k < 1} p(1-k)\left(1 - \frac{p}{\alpha - \alpha k + k}\right).$$

  The profit is concave in p. The first order derivative w.r.t p is:

$$\frac{\partial \pi_S}{\partial p} = \frac{(1-k)(\alpha + (1-\alpha)k - 2p)}{\alpha - \alpha k + k}.$$

  From FOC, the uncontrained optimizer is $\bar{p} = \frac{\alpha + (1-\alpha)k}{2}$. We have:

$$\begin{aligned} \bar{p} \geq (1+\lambda)\alpha \quad &\iff \quad k \geq \frac{2\alpha\lambda + \alpha}{1-\alpha}, \\ \frac{2\alpha\lambda + \alpha}{1-\alpha} < 1 \quad &\iff \quad \alpha(\lambda+1) < \frac{1}{2}. \end{aligned}$$

  We get two subcases:

  - Case 2-i: $\alpha(\lambda+1) < \frac{1}{2}$.
    Then $\frac{2\alpha\lambda+\alpha}{1-\alpha} < 1$.

    * Case 2-i-a: $0 \leq k < \frac{2\alpha\lambda+\alpha}{1-\alpha}$.
      Then $\bar{p} < (1+\lambda)\alpha$. As such, we have the corner solution $p_S^* = \alpha(\lambda+1)$. The firm's profit maximization problem becomes:

$$\max_{0 \leq k < \frac{2\alpha\lambda+\alpha}{1-\alpha}} \pi_S = \max_{0 \leq k < \frac{2\alpha\lambda+\alpha}{1-\alpha}} \alpha(\lambda+1)(1-k)\left(1 - \frac{\alpha(\lambda+1)}{\alpha - \alpha k + k}\right).$$

      We have:

$$\frac{\partial \pi_S}{\partial k} = -\frac{\alpha(\lambda+1)\left(-\alpha\lambda - \alpha + (\alpha - \alpha k + k)^2\right)}{(\alpha - \alpha k + k)^2}.$$

      Solving the unconstrained equation $\frac{\partial \pi_S}{\partial k} = 0$, we obtain two candidate solutions:

$$k_1 = \frac{-\alpha - \sqrt{\alpha(\lambda+1)}}{1-\alpha} < 0 < k_2 = \frac{-\alpha + \sqrt{\alpha(\lambda+1)}}{1-\alpha}.$$

      π_S is decreasing in k on $(-\infty, k_1)$, increasing on (k_1, k_2), and then decreasing on (k_2, ∞). Comparing k_2 and $\frac{2\alpha\lambda+\alpha}{1-\alpha}$, we get two subcases:

      · Case 2-i-a-I: $\frac{1}{4} < \alpha(\lambda+1) < \frac{1}{2}$.
        Then, $k_2 < \frac{2\alpha\lambda+\alpha}{1-\alpha}$. Thus, we get the interior solution $k_S^* = k_2$ and $\pi_S^* =$

$\frac{\alpha(\lambda+1)\left(1+\alpha((1-\alpha)\lambda+\alpha)-2(1-\alpha)\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)^2}$.

- Case 2-i-a-II: $0 < \alpha(\lambda + 1) \leq \frac{1}{4}$.
  Then, $k_2 \geq \frac{2\alpha\lambda+\alpha}{1-\alpha}$. Thus, π_S is increasing in k on the entire region and this case gets dominated by case 2-i-b-II.

∗ Case 2-i-b: $\frac{2\alpha\lambda+\alpha}{1-\alpha} \leq k < 1$.
In this case, $\bar{p} \geq (1+\lambda)\alpha$. Thus, we have the interior solution $p_S^* = \bar{p} = \frac{\alpha+(1-\alpha)k}{2}$. The firm's profit maximization problem becomes:

$$\max_{\frac{2\alpha\lambda+\alpha}{1-\alpha} \leq k < 1} \pi_S = \max_{\frac{2\alpha\lambda+\alpha}{1-\alpha} \leq k < 1} \frac{(1-k)(\alpha + k(1-\alpha))}{4}.$$

The profit function is concave in k. We have:

$$\frac{\partial \pi_S}{\partial k} = \frac{1}{4}(1 - 2\alpha - 2(1-\alpha)k).$$

Solving the unconstrained equation $\frac{\partial \pi_S}{\partial k} = 0$, we obtain the candidate solution $\bar{k}_S = \frac{1-2\alpha}{2(1-\alpha)} < 1$. We also have:

$$\frac{2\alpha\lambda + \alpha}{1-\alpha} \leq \bar{k}_S \quad \Longleftrightarrow \quad \alpha(\lambda+1) \leq \frac{1}{4}.$$

We get two subcases:

- Case 2-i-b-I: $\frac{1}{4} < \alpha(\lambda+1) < \frac{1}{2}$.
  Then, $\frac{2\alpha\lambda+\alpha}{1-\alpha} > \bar{k}_S$. As such, we get the corner solution $k^* = \frac{2\alpha\lambda+\alpha}{1-\alpha}$. Substituting k_S^*, we obtain $p_S^* = \alpha(\lambda+1)$ and $\pi_S^* = \frac{\alpha(\lambda+1)(1-2\alpha(\lambda+1))}{2(1-\alpha)}$.

- Case 2-i-b-II: $\alpha(\lambda+1) \leq \frac{1}{4}$.
  Then, $\frac{2\alpha\lambda+\alpha}{1-\alpha} > \bar{k}_S$ and we get the interior solution $k_S^* = \bar{k}_S = \frac{1-2\alpha}{2(1-\alpha)}$. Substituting k_S^*, we obtain $p_S^* = \frac{1}{4}$ and $\pi_S^* = \frac{1}{16(1-\alpha)}$.

Comparing Cases 2-i-a and 2-i-b, we get:

∗ If $\alpha(\lambda+1) \leq \frac{1}{4}$, $p_S^* = \frac{1}{4}$, $k_S^* = \frac{1-2\alpha}{2(1-\alpha)}$, $\pi_S^* = \frac{1}{16-16\alpha}$.

∗ If $\frac{1}{4} < \alpha(\lambda+1) < \frac{1}{2}$, then $p_S^* = \alpha(\lambda+1)$ under both 2-i-a-I and 2-i-b-I. Comparing the profits directly, it can be shown that Case 2-i-a-I dominates. Thus, $k_S^* = k_2$ and $\pi_S^* = \frac{\alpha(\lambda+1)\left(1+\alpha((1-\alpha)\lambda+\alpha)-2(1-\alpha)\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)^2}$.

[Comparison between S and *CE-PL*]

Under both cases (i.e., when $\alpha(\lambda+1) < \frac{1}{2}$), we get $0 < \alpha < 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$. Therefore, in this region $\pi^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$. We have two regions to compare:

* If $\alpha(\lambda+1) \leq \frac{1}{4}$, then $\pi^*_S = \frac{1}{16-16\alpha}$. It can be shown that:

$$\pi^*_S > \pi^*_{CE-PL} \iff \begin{cases} \alpha(\lambda+1) \leq \frac{1}{4}, \text{ and} \\ 32\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}+1 > \alpha(16\alpha(\lambda+1)(2\lambda+1)+16\lambda+17). \end{cases} \tag{C.2}$$

When S dominates *CE-PL*, we have $SW^*_S = k^*_S \int_0^1 (1+\lambda)\theta d\theta + (1-k^*_S)\int^1_{\frac{p^*_S}{\alpha-\alpha k^*_S+k^*_S}} \theta d\theta = \frac{4\lambda+7-8\alpha(\lambda+1)}{16(1-\alpha)}$.

Let us better understand the region characterized under condition (C.2).

· If $0 < \alpha \leq \frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}$, then $\alpha(16\alpha(\lambda+1)(2\lambda+1)+16\lambda+17)-1 \leq 0$. Then, the inequality $32\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}+1 > \alpha(16\alpha(\lambda+1)(2\lambda+1)+16\lambda+17)$ is always satisfied.

· If $\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}} < \alpha \leq \frac{1}{4(\lambda+1)}$,[C-1] then $\alpha(16\alpha(\lambda+1)(2\lambda+1)+16\lambda+17)-1 > 0$. In this region, we have:

$$\begin{aligned} &32\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}+1 > \alpha(16\alpha(\lambda+1)(2\lambda+1)+16\lambda+17) \\ &\quad\iff \Gamma(\alpha) \triangleq \alpha(32\alpha(\lambda+1)(8\alpha(\lambda+1)-6\lambda-7)+32\lambda+33)-1 > 0. \end{aligned}$$

Solving $\frac{\partial\Gamma(\alpha)}{\partial\alpha} = 0$, we get two solutions:

$$\tilde{\alpha}_1 = \frac{2\lambda(6\lambda+13)+14-(\lambda+1)\sqrt{48\lambda(3\lambda+5)+97}}{48(\lambda+1)^2},$$
$$\tilde{\alpha}_2 = \frac{2\lambda(6\lambda+13)+14+(\lambda+1)\sqrt{48\lambda(3\lambda+5)+97}}{48(\lambda+1)^2}.$$

It can be easily shown that $0 < \frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}} < \tilde{\alpha}_1 < \frac{1}{4(\lambda+1)} < \tilde{\alpha}_2$, $\Gamma(\tilde{\alpha}_1) > 0$, $\Gamma\left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}\right) > 0$, $\Gamma\left(\frac{1}{4(\lambda+1)}\right) < 0$, $\Gamma(\tilde{\alpha}_2) < 0$. In terms of monotonicity, $\Gamma(\alpha)$ is increasing on $\left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}, \tilde{\alpha}_1\right)$ and then decreasing on $\left(\tilde{\alpha}_1, \frac{1}{4(\lambda+1)}\right]$. Therefore, there exits a unique $\alpha^\ddagger \in \left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}, \frac{1}{4(\lambda+1)}\right)$, such that $\Gamma(\alpha^\ddagger) = 0$. Thus, when $\alpha \in$

[C-1] We also verified that $\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}} < \frac{1}{4(\lambda+1)}$ to make sure this region exists.

$\left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}, \alpha^{\ddagger}\right)$, S dominates *CE-PL*, and when $\alpha \in \left[\alpha^{\ddagger}, \frac{1}{4(\lambda+1)}\right)$, *CE-PL* dominates S.

Moreover, the range for $\alpha^{\ddagger}$ can be further narrowed to:

$$\alpha^{\ddagger} \in \left(\tilde{\alpha}_1, \frac{1}{4(\lambda+1)}\right) \subset \left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}, \frac{1}{4(\lambda+1)}\right].$$

Thus, putting the two subregions together, S dominates *CE-PL* when $0 < \alpha < \alpha^{\ddagger}$ and *CE-PL* dominates S when $\alpha \in \left[\alpha^{\ddagger}, \frac{1}{4(\lambda+1)}\right)$.

* If $\frac{1}{4} \le \alpha(\lambda+1) < \frac{1}{2}$, it can be shown that:

$$\begin{aligned}\pi_S^* &= \frac{\alpha(\lambda+1)\left(1+\alpha((1-\alpha)\lambda+\alpha)-2(1-\alpha)\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)^2} \\ &< \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2} = \pi^*_{CE-PL}.\end{aligned}$$

– Case 2-ii: $\alpha(\lambda+1) \ge \frac{1}{2}$.
Then $\frac{2\alpha\lambda+\alpha}{1-\alpha} \ge 1 > k$. As such $\bar{p} < (1+\lambda)\alpha$. Thus, we have the corner solution $p_S^* = \alpha(\lambda+1)$. Following the same steps as Case 2-i, we get threshold values $k_1 < 0 < k_2$. Comparing k_2 with 1, we have two subcases:

* Case 2-ii-a: $\frac{1}{2} \le \alpha(\lambda+1) < 1$.
Then $k_S^* = k_2 = \frac{-\alpha+\sqrt{\alpha(\lambda+1)}}{1-\alpha}$ and $\pi_S^* = \frac{\alpha(\lambda+1)\left(1+\alpha((1-\alpha)\lambda+\alpha)-2(1-\alpha)\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)^2}$. It can be shown that:

$$\pi_S^* < \begin{cases} \frac{\alpha(\lambda+1)(2\alpha\lambda+\alpha+1)-2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}}{(1-\alpha)^2} = \pi^*_{CE-PL} & \text{, if } \frac{1}{2(\lambda+1)} \le \alpha < 5+8\lambda-4\sqrt{(\lambda+1)(4\lambda+1)}, \\ \frac{1}{4}\alpha(\lambda+1) = \pi^*_{CE-PL} & \text{, if } 5+8\lambda-4\sqrt{(\lambda+1)(4\lambda+1)} \le \alpha < \frac{1}{\lambda+1} \end{cases}$$

Thus, S is dominated by *CE-PL* in this region.

* Case 2-ii-b: If $\alpha(\lambda+1) \ge 1$.
Then, $k_S^* = 1$ and $\pi_S^* = 0$. This case is clearly suboptimal - in this region S is obviously dominated by *CE-PL* since the latter generates non-zero profit when $\alpha \ge \frac{1}{\lambda+1}$.[C-2] □

Proof of Proposition 2.
When $\underline{\alpha \ge 1}$, by directly comparing profits and social welfare values from Propositions C.1-C.4, it can be easily seen that *CE-PL* is always the dominant strategy for the firm, whereas *TLF* is always

[C-2] We have $5+8\lambda-4\sqrt{(\lambda+1)(4\lambda+1)} < \frac{1}{\lambda+1} \le \alpha$. In this region, $\pi^*_{CE-PL} = \frac{1}{4}\alpha(\lambda+1)$.

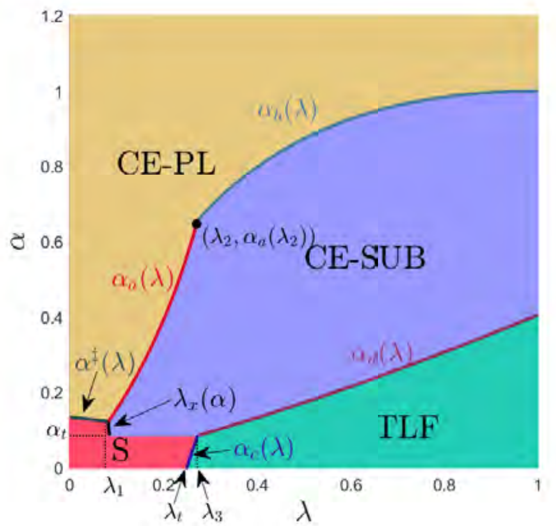

Figure C.1: Individual Depreciation Scenario - Optimal Business Model - Marked Boundaries

the strategy that yields the highest social welfare.

The bulk of the proof, below, is addressing the considerably more complex case $\underline{0 < \alpha < 1}$.

Let us define:

$$\alpha_1(\lambda) \triangleq \begin{cases} \alpha^{\ddagger}(\lambda) & , \text{if } 0 \leq \lambda < \lambda_1, \\ \alpha_a(\lambda) & , \text{if } \lambda_1 \leq \lambda < \lambda_2, \\ \alpha_b(\lambda) & , \text{if } \lambda_2 \leq \lambda \leq 1, \end{cases}$$

and

$$\alpha_2(\lambda) \triangleq \begin{cases} \alpha_c(\lambda) & , \text{if } \frac{1}{4} \leq \lambda < \lambda_3, \\ \alpha_d(\lambda) & , \text{if } \lambda_3 \leq \lambda \leq 1, \end{cases}$$

where $\alpha^{\ddagger}(\lambda)$ was defined in Prop C.4, and functions $\alpha_a(\cdot)$, $\alpha_b(\cdot)$, $\alpha_c(\cdot)$, $\alpha_d(\cdot)$, as well as constant thresholds λ_1, λ_2, and λ_3 are defined and further analyzed below. For ease of identification, Figure C.1 contains the illustration of these boundaries and thresholds (this is a more detailed version of Figure 2 from the main body).

- **Monotonicity of $\alpha^{\ddagger}(\lambda)$.**
  As discussed in the text and proof of Prop. C.4, $\alpha^{\ddagger}(\lambda)$ represents the boundary between the regions where S dominates *CE-PL* and the region where *CE-PL* dominates S (i.e., the region in which S, under optimality, requires $k^* = 0$, effectively defaulting to *CE-PL*). We have shown that $\alpha^{\ddagger}(\lambda)$ exists, it is unique (thus, it is well defined for all $\lambda \in (0, 1)$) and it satisfies:

$$\frac{\alpha^{\ddagger}(\lambda)(\lambda+1)\left(2\alpha^{\ddagger}(\lambda)\lambda + \alpha^{\ddagger}(\lambda) + 1 - 2\sqrt{\alpha^{\ddagger}(\lambda)(\lambda+1)(\alpha^{\ddagger}(\lambda)\lambda+1)}\right)}{(1-\alpha^{\ddagger}(\lambda))^2} - \frac{1}{16 - 16\alpha^{\ddagger}(\lambda)} = 0.$$

Define:

$$\Psi_{\ddagger}(\alpha,\lambda) \triangleq \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2} - \frac{1}{16(1-\alpha)}.$$

We have $\Psi_{\ddagger}(\alpha^{\ddagger}(\lambda),\lambda)=0$. At the same time, for all α, $\lambda \in (0,1)$, it can be shown that:

$$\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \alpha} = \frac{\left(\alpha(8\lambda+7)^2+16\lambda+15\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} - 48\alpha(\lambda+1)^2 - 16\alpha^2(4\lambda+1)(\lambda+1)^2}{16(1-\alpha)^3\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}} > 0, \quad \text{(C.3)}$$

$$\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \lambda} = \frac{\alpha\left(4\alpha\lambda+3\alpha+1-\frac{\alpha(\lambda+1)(4\alpha\lambda+\alpha+3)}{\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}}\right)}{(1-\alpha)^2} > 0. \quad \text{(C.4)}$$

Therefore, $\frac{\partial \alpha^{\ddagger}(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \alpha}} < 0$. Hence, $\alpha^{\ddagger}(\lambda)$ is decreasing in λ.

- **Definition of λ_1, λ_2, and $\alpha_a(\lambda)$. Monotonicity of $\alpha_a(\lambda)$.**
  We know that when $\alpha \geq \alpha^{\ddagger}$, *CE-PL* dominates *S*. In this same region ($\underline{\alpha \geq \alpha^{\ddagger}}$), let us further compare profits under *CE-PL* and *CE-SUB* strategies.

  - First, the following two inequalities can be easily shown:

  $$\frac{\alpha}{(\sqrt{\alpha}+1)^2} < \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}, \quad \forall \lambda \in (0,1),\ \alpha \in (\lambda,1),$$

  $$\frac{\alpha}{(\sqrt{\alpha}+1)^2} < \frac{1}{4}\alpha(1+\lambda), \quad \forall \lambda \in (0,1),\ \max\{5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}, \lambda\} < \alpha < 1.$$

  Thus, given that $\lambda < \alpha^{\dagger}$, we see that in the region $\alpha^{\dagger} < \alpha \leq 1$ (third case in Prop. C.2) we have $\pi^*_{CE-SUB} < \pi^*_{CE-PL}$.

  - We further compare π^*_{CE-PL} under the first case in Prop. C.1 and π^*_{CE-SUB} under the second case in Prop. C.2. As we stay within region $\alpha \geq \alpha^{\ddagger}$, we look at the parameter region at the intersection among regions $\alpha \geq \alpha^{\ddagger}$, $\lambda < \alpha \leq \alpha^{\dagger}$, and $\alpha < 5+8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$. Since $\alpha^{\ddagger} < 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$, it can immediately follows that this is a non-empty region. In this region, define the difference between optimal profits under *CE-SUB* and *CE-PL* as:

  $$\Psi_a(\alpha,\lambda) \triangleq p_b\left(1-\frac{p_b}{\lambda}+1-\frac{p_b}{1+p_b-\frac{p_b}{\alpha}}\right) - \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}.$$

  Let's next try to understand the monotonicity of $\Psi_a(\alpha,\lambda)$ with respect to α and λ. After taking derivatives and applying the Envelope theorem with respect to π^*_{CE-SUB}, given that $p_b(\alpha, \lambda)$ represents the maximizing price for *CE-SUB*, we obtain:

$$\begin{aligned}
\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha} =& \frac{2\alpha(\lambda+1)\left(2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}-2\alpha\lambda+\alpha+1\right)}{(1-\alpha)^3} \\
&- \frac{(\lambda+1)\left(\alpha\left(-\frac{(\lambda+1)(4\alpha\lambda+3)}{\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}}+4\lambda+2\right)+1\right)}{(1-\alpha)^2} + \frac{p_b^3}{(\alpha-(1-\alpha)p_b)^2}, \\
\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \lambda} =& \frac{p_b^2}{\lambda^2} - \frac{\alpha\left(-\frac{\alpha(\lambda+1)(4\alpha\lambda+\alpha+3)}{\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}}+2\alpha(\lambda+1)+2\alpha\lambda+\alpha+1\right)}{(1-\alpha)^2}.
\end{aligned}$$

We know from the proof of Prop. C.2 that $p_b \in \left(\frac{\lambda}{2}, \lambda\right)$. Using these additional bounds on p_b, it is easy to get $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \lambda} > 0$. Therefore, for any given α, when we increase λ there can be at most one crossing point that separates the optimality regions for *CE-SUB* and *CE-PL*, and, moreover, the crossing (if it exists) can only be from *CE-PL* to *CE-SUB* as λ increases in this region of the parameter space.

Next, let's check the sign of $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha}$. Bringing all the terms to a common denominator, we can write $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha} = \frac{q_1}{q_2}$, where:

$$\begin{aligned}
q_1 =& p_b^3(1-\alpha)^3\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} \\
&+ p_b^2\alpha(1-\alpha)^2(\lambda+1)^2\left((\lambda+1)(4\alpha^R 2\lambda+\alpha^2+3\alpha)+(\alpha(4\lambda+3)+1)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right) \\
&- 2p_b\alpha^2(1-\alpha)(\lambda+1)^2\left((\lambda+1)(4\alpha^2\lambda+\alpha^2+3\alpha)-(\alpha(4\lambda+3)+1)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right) \\
&+ \alpha^3(\lambda+1)^2\left((\lambda+1)(4\alpha^2\lambda+\alpha^2+3\alpha)-(\alpha(4\lambda+3)+1)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right), \\
q_2 =& (1-\alpha)^3\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}(\alpha+\alpha p_b-p_b)^2 > 0.
\end{aligned}$$

Therefore, the sign of $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha}$ is the same as the sign of the numerator, q_1.

Recall from Prop. C.2 that p_b is the unique solution to the equation $G_{SUB,b}(p) = 2\alpha^2\lambda - 2(1-\alpha)^2p^3 - (1-\alpha)p^2(\alpha(\lambda-4)-2\lambda) - 2\alpha p(\alpha(1-\lambda)+2\lambda) = 0$. We use this property of p_b (i.e., $G_{SUB,b}(p_b) = 0$) to reduce the expression of q_1 from a cubic polynomial in p_b to a quadratic one, as follows:

$$\begin{aligned}
q_1 =& \alpha(\lambda+1) \\
&\times \Bigg(p_b^2\frac{1}{2}(1-\alpha)^2\left(2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)-\left(\alpha\left(8\lambda^2+15\lambda+2\right)+2\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right) \\
&\quad + p_b(1-\alpha)\alpha\left(\left(\alpha\left(8\lambda^2+15\lambda+5\right)+2\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}-2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)\right) \\
&\quad + \alpha^2\left(2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)-\left(\alpha\left(4\lambda^2+8\lambda+3\right)+1\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)\Bigg).
\end{aligned}$$

Denote:

$$A \triangleq \frac{1}{2}(1-\alpha)^2\left(2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3) - \left(\alpha\left(8\lambda^2+15\lambda+2\right)+2\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right),$$
$$B \triangleq (1-\alpha)\alpha\left(\left(\alpha\left(8\lambda^2+15\lambda+5\right)+2\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} - 2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)\right),$$
$$C \triangleq \alpha^2\left(2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3) - \left(\alpha\left(4\lambda^2+8\lambda+3\right)+1\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right).$$

Then $\frac{q_1}{\alpha(\lambda+1)} = Ap_b^2 + Bp_b + C$. Define quadratic function $H_{SUB,PL}(p) \triangleq Ap^2 + Bp + C$. In this range of the parameter space, it can be shown that:

$$\begin{aligned} B^2 - 4AC = (1-\alpha)^2\alpha^4(\lambda+1) \\ \times\Big(2\left(\lambda^2-\lambda-2\right)(4\alpha\lambda+\alpha+3)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} \\ -((\alpha\lambda+1)(\alpha(\lambda(\lambda(8\lambda-1)-28)-13)+2(\lambda-2)))\Big) > 0. \end{aligned}$$

Hence, there are two real solutions of $H_{SUB,PL}(p) = 0$, namely:

$$p_{H1} = -\frac{B+\sqrt{B^2-4AC}}{2A} \quad \text{and} \quad p_{H2} = \frac{-B+\sqrt{B^2-4AC}}{2A}.$$

It can be shown that $p_{H1} < p_{H2} < \frac{\alpha}{1-\alpha}$. Recall that p_b is the unique solution of $G_{SUB,b}(p) = 0$. Moreover, from the proof of Prop. C.2, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞).

It can be proved directly that $G_{SUB,b}(p_{H1}) > 0 = G_{SUB,b}(p_b) > G_{SUB,b}(p_{H2})$. Hence, $p_{H1} < p_b < p_{H2}$.

Furthermore, it can be shown that $A > 0$, which indicates that $H_{SUB,PL}(p)$ is convex. Therefore, $H_{SUB,PL}(p_b) < 0$. Thus, $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha} < 0$. Therefore, for any given λ, when we increase α, there can be at most one crossing point that separates the optimality regions for *CE-SUB* and *CE-PL*, and, moreover, the crossing (if it exists) can be only from *CE-SUB* to *CE-PL* as α increases.

So far, we proved that a threshold (crossing) boundary between optimality regions for *CE-SUB* and *CE-PL* within this particular region of the parameter space (at the intersection among regions $\alpha \geq \alpha^\ddagger$, $\lambda < \alpha \leq \alpha^\dagger$, and $\alpha < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$) is unique for every λ and for every α (i.e., if we look vertically or horizontally), *if it exists*. Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space.

We look at two particular delimiting boundaries for this region, namely $\alpha = \alpha^\dagger$ and $\alpha = \lambda$ and examine the sign of $\Psi_a(\alpha, \lambda)$ along these boundaries.

* On the boundary $\alpha = \alpha^\dagger(\lambda)$, since we are under condition $\alpha < 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$

< 1, by definition of $\alpha^\dagger$, we obtain:

$$\Psi_a(\alpha,\lambda)\Big|_{\alpha=\alpha^\dagger(\lambda)} = \frac{\alpha}{(\sqrt{\alpha}+1)^2} - \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2} < 0.$$

* On the boundary $\alpha = \lambda$, we obtain:

$$\Psi_a(\alpha,\lambda)\Big|_{\alpha=\lambda} = p_b\left(2-\frac{p_b}{\lambda}-\frac{p_b}{1+p_b-\frac{p_b}{\lambda}}\right) - \frac{\lambda(\lambda+1)\left(2\lambda^2+\lambda+1-2\sqrt{\lambda(\lambda+1)(\lambda^2+1)}\right)}{(1-\lambda)^2}.$$

We point out that we could have written the profit for *CE-SUB* in terms of p_a at the boundary when $\alpha = \lambda$ - however, on that boundary, whether we write the profit in terms of p_a or p_b, we obtain the same profit because on that particular line, $p_a = p_b$ (as they satisfy the same implicit equation). Bringing all the terms to a common denominator, we can write $\Psi_a(\alpha,\lambda)\,|_{\alpha=\lambda} = \frac{q_3}{q_4}$, where:

$$\begin{aligned} q_3 =&(1-\lambda)^3p_b^3 - (3-\lambda)(1-\lambda)^2\lambda p_b^2 \\ &+ (1-\lambda)\lambda^2\left(2\lambda^3+3\lambda^2+3-2\sqrt{\lambda(\lambda+1)^3\left(\lambda^2+1\right)}\right)p_b \\ &- \lambda^2\left(2\lambda^4+3\lambda^3+2\lambda^2-2\sqrt{\lambda^3(\lambda+1)^3\left(\lambda^2+1\right)}+\lambda\right), \\ q_4 =&(1-\lambda)^2\lambda(\lambda+\lambda p_b - p_b) > 0. \end{aligned}$$

Therefore, the sign of $\Psi_a(\alpha,\lambda)\,|_{\alpha=\lambda}$ is the same as the sign of the numerator, q_3.

We use $G_{SUB,b}(p_b) = 0$ to reduce the expression of q_3 from a cubic polynomial in p_b to a quadratic one, as follows:

$$\begin{aligned} q_3 =&\frac{\lambda^2}{2} \\ &\times\Big((1-\lambda)^2p_b^2 + 2\left(1-\lambda^2\right)\left(2\lambda^2-2\sqrt{\lambda(\lambda+1)\left(\lambda^2+1\right)}+\lambda\right)p_b \\ &\quad +4\lambda(\lambda+1)\sqrt{\lambda(\lambda+1)\left(\lambda^2+1\right)} - 2\lambda^2\left(2\lambda^2+3\lambda+3\right)\Big). \end{aligned}$$

Denote:

$$\begin{aligned} D &\triangleq (1-\lambda)^2, \\ E &\triangleq 2\left(1-\lambda^2\right)\left(2\lambda^2-2\sqrt{\lambda(\lambda+1)\left(\lambda^2+1\right)}+\lambda\right), \\ F &\triangleq 4\lambda(\lambda+1)\sqrt{\lambda(\lambda+1)\left(\lambda^2+1\right)} - 2\lambda^2\left(2\lambda^2+3\lambda+3\right). \end{aligned}$$

Then $\frac{2q_3}{\lambda^2} = Dp_b^2 + Ep_b + F$. Define quadratic function $\tilde{H}_{SUB,PL}(p) \triangleq Dp^2+Ep+F$. In this range of the parameter space, it can be shown that $E^2 - 4DF > 0$. Hence, there are two real solutions to the equation $\tilde{H}_{SUB,PL}(p) = 0$, namely:

$$p_{\tilde{H}1} = \frac{-E-\sqrt{E^2-4DF}}{2D} \quad \text{and} \quad p_{\tilde{H}2} = \frac{-E+\sqrt{E^2-4DF}}{2D}.$$

It can be shown that $\frac{\lambda}{2} < p_{\tilde{H}1} < \lambda < p_{\tilde{H}2}$. Recall that p_b is the unique solution of $G_{SUB,b}(p) = 0$. Moreover, from the proof of Prop. C.2, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(p_{\tilde{H}1}) < 0 = G_{SUB,b}(p_b)$. Hence, $p_b < p_{\tilde{H}1} < p_{\tilde{H}2}$. Furthermore, it can be shown that $D > 0$, which indicates that $\tilde{H}_{SUB,PL}(p)$ is convex. Therefore, $\tilde{H}_{SUB,PL}(p_b) > 0$. Hence, in this region of the parameter space:

$$\Psi_a(\alpha, \lambda)\Big|_{\alpha=\lambda} > 0.$$

Thus, $\Psi_a(\alpha, \lambda)\Big|_{\alpha=\alpha^\dagger(\lambda)} < 0$ and $\Psi_a(\alpha, \lambda)\Big|_{\alpha=\lambda} > 0$. Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_a(\lambda)}$, which separates the optimality regions for *CE-SUB* and *CE-PL*, and which falls between boundaries $\alpha = \alpha^\dagger$ and $\alpha = \lambda$. It satisfies:

$$\frac{\alpha_a(\lambda)(\lambda+1)\left(2\alpha_a(\lambda)\lambda + \alpha_a(\lambda) + 1 - 2\sqrt{\alpha_a(\lambda)(\lambda+1)(\alpha_a(\lambda)\lambda+1)}\right)}{(1-\alpha_a(\lambda))^2} = p_b\left(1 - \frac{p_b}{\lambda} + 1 - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha_a(\lambda)}}\right).$$

Since existence and uniqueness are satisfied, $\alpha_a(\lambda)$ is properly defined as a function. Moreover, since $\Psi_a(\alpha_a(\lambda), \lambda) = 0$, by differentiation w.r.t. λ, we obtain $\frac{\partial \alpha_a(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha}} > 0$. Hence, $\alpha_a(\lambda)$ is increasing.

Since both boundaries $\alpha^\ddagger$ and $\alpha = 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$ are decreasing in λ and $\alpha_a(\lambda)$ is increasing in lambda and strictly between the lines $\alpha^\dagger$ and $\alpha = \lambda$, then there exists a unique intersection point between $\alpha_a(\lambda)$ and $\alpha^\ddagger$, and a unique intersection point between $\alpha_a(\lambda)$ and $\alpha = 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$.

* Define $\{\lambda_1, \alpha^\ddagger(\lambda_1)\}$ as the unique intersection between $\alpha_a(\lambda)$ and $\alpha^\ddagger$. Then, λ_1 satisfies:

$$\frac{1}{16(1-\alpha^\ddagger(\lambda_1))} = \frac{\alpha^\ddagger(\lambda_1)(\lambda_1+1)(2\alpha^\ddagger(\lambda_1)\lambda_1 + \alpha^\ddagger(\lambda_1) + 1) - 2\sqrt{\alpha^\ddagger(\lambda_1)^3(\lambda_1+1)^3(\alpha^\ddagger(\lambda_1+1)}}{(1-\alpha^\ddagger)^2}$$
$$= p_b\left(1 - \frac{p_b}{\lambda_1} + 1 - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha^\ddagger(\lambda_1)}}\right).$$

  More precisely, at $\{\lambda_1, \alpha^\ddagger(\lambda_1)\}$, we have:

$$\pi^*_{CE-PL}(\lambda_1, \alpha^\ddagger(\lambda_1)) = \pi^*_{CE-SUB}(\lambda_1, \alpha^\ddagger(\lambda_1)) = \pi^*_S(\lambda_1, \alpha^\ddagger(\lambda_1)).$$

* Define $\{\lambda_2, 5+8\lambda_2-4\sqrt{(1+\lambda_2)(1+4\lambda_2)}\}$ as the unique intersection between $\alpha_a(\lambda)$ and $\alpha = 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$. Then, λ_2 satisfies:

$$\frac{\left(5 + 8\lambda_2 - 4\sqrt{(1+\lambda_2)(1+4\lambda_2)}\right)(1+\lambda_2)}{4} = p_b\left(1 - \frac{p_b}{\lambda_2} + 1 - \frac{p_b}{1 + p_b - \frac{p_b}{5+8\lambda_2-4\sqrt{(1+\lambda_2)(1+4\lambda_2)}}}\right).$$

It immediately follows that $\lambda_1 < \lambda_2$ and $\alpha_a(\lambda)$ is properly defined and increasing on $\lambda \in [\lambda_1, \lambda_2]$. We show $\alpha_a(\lambda)$ in Figure C.2.

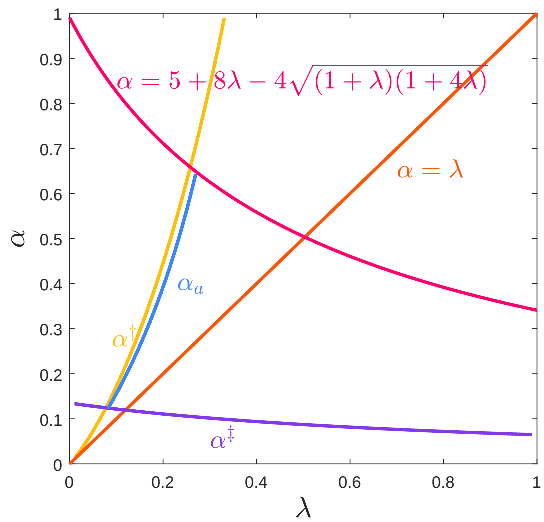

Figure C.2: $\alpha_a(\lambda)$

- **Definition and monotonicity of $\alpha_b(\lambda)$.**
  We further compare the second case under *CE-PL* and the second case under *CE-SUB* at the intersection among regions $5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)} \le \alpha < 1$ and $\lambda < \alpha \le \alpha^\dagger$. Denote the profit difference between *CE-SUB* and *CE-PL* in this region as:

$$\Psi_b(\alpha, \lambda) \triangleq p_b\left(1 - \frac{p_b}{\lambda} + 1 - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha}}\right) - \frac{\alpha(1+\lambda)}{4}.$$

Then, using Envelope theorem (since p_b maximizes π_{CE-SUB}), we have :

$$\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \alpha} = \frac{p_b^3}{(\alpha-(1-\alpha)p_b)^2} - \frac{1+\lambda}{4},$$
$$\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \lambda} = \frac{p_b^2}{\lambda^2} - \frac{\alpha}{4} > \frac{(\frac{\lambda}{2})^2}{\lambda^2} - \frac{\alpha}{4} > 0.$$

Next, let's check the sign of $\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \alpha}$. Bringing all the terms to a common denominator, we can write $\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \alpha} = \frac{q_5}{q_6}$, where:

$$q_5 \triangleq -\alpha^2\lambda - \alpha^2 + 4p_b^3 + p_b^2\left(-\alpha^2\lambda - \alpha^2 + 2\alpha\lambda + 2\alpha - \lambda - 1\right) + p_b\left(-2\alpha^2\lambda - 2\alpha^2 + 2\alpha\lambda + 2\alpha\right),$$
$$q_6 \triangleq 4(\alpha + \alpha p_b - p_b)^2 > 0.$$

Thus, the sign of $\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \alpha}$ is the same as the sign of the numerator, q_5. We use $G_{SUB,b}(p_b) = 0$ to reduce the expression of q_5 from a cubic polynomial in p_b to a quadratic one, as follows:

$$\begin{aligned} q_5 = &-p_b^2(1-\alpha)(\alpha(-(3-\alpha)\alpha(\lambda+1)+\lambda+11)+3\lambda-1) \\ &+ p_b 2\alpha(\alpha(-(3-\alpha)\alpha(\lambda+1)+\lambda+5)+3\lambda-1) \\ &+ \alpha^2(-(2-\alpha)\alpha(\lambda+1)-3\lambda+1). \end{aligned}$$

Denote:

$$\begin{aligned} J &\triangleq -(1-\alpha)(\alpha(-(3-\alpha)\alpha(\lambda+1)+\lambda+11)+3\lambda-1), \\ K &\triangleq 2\alpha(\alpha(-(3-\alpha)\alpha(\lambda+1)+\lambda+5)+3\lambda-1), \\ L &\triangleq \alpha^2(-(2-\alpha)\alpha(\lambda+1)-3\lambda+1). \end{aligned}$$

Then, $q_5 = Jp_b^2 + Kp_b + L$. Define the quadratic function $\bar{H}_{SUB,PL}(p) \triangleq Jp^2 + Kp + L$. In this range of the parameter space, it can be shown that $K^2 - 4JL > 0$. Hence, there are two real solutions to the equation $\bar{H}_{SUB,PL}(p) = 0$, namely:

$$p_{\bar{H}1} = \frac{-K - \sqrt{K^2 - 4JL}}{2J} \quad \text{and} \quad p_{\bar{H}2} = \frac{-K + \sqrt{K^2 - 4JL}}{2J}.$$

It can be shown that $p_{\bar{H}2} < \lambda < p_{\bar{H}1}$. Recall that p_b is the unique solution of $G_{SUB,b}(p) = 0$. Moreover, from the proof of Prop. C.2, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(p_{\bar{H}2}) > 0 = G_{SUB,b}(p_b)$. Hence, $p_{\bar{H}2} < p_b < \lambda < p_{\bar{H}1}$. Furthermore, it can be shown that $J < 0$, which indicates that $\bar{H}_{SUB,PL}(p)$ is concave. Therefore, $\bar{H}_{SUB,PL}(p_b) > 0$. Hence, in this region of the parameter space:

$$\frac{\partial \Psi_b(\alpha, \lambda)}{\partial \alpha} > 0.$$

So far, we proved that a threshold (crossing) boundary between optimality regions for *CE-SUB* and *CE-PL* within this particular region of the parameter space (at the intersection among regions $5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)} \leq \alpha < 1$ and $\lambda < \alpha \leq \alpha^\dagger$) is unique for every λ and for every α (i.e., if we look vertically or horizontally), *if it exists.*

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = \alpha^\dagger$ and $\alpha = \lambda$ (boundary in limit) and examine the sign of $\Psi_b(\alpha, \lambda)$ along these boundaries.

- On the boundary $\alpha = \alpha^\dagger(\lambda)$, since we are under condition $5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)} \leq \alpha < 1$, by definition of $\alpha^\dagger$, we obtain:

$$\Psi_b(\alpha, \lambda)\Big|_{\alpha = \alpha^\dagger(\lambda)} = \frac{\alpha}{(\sqrt{\alpha}+1)^2} - \frac{\alpha(1+\lambda)}{4} < 0.$$

- On the boundary $\alpha = \lambda$ (boundary in limit), we obtain:

$$\Psi_b(\alpha, \lambda)\Big|_{\alpha = \lambda} = p_b \left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1 + p_b - \frac{p_b}{\lambda}}\right) - \frac{\lambda(1+\lambda)}{4}.$$

  Again, we remind the reader we could have written the profit for $CE-SUB$ in terms of p_a at the boundary when $\alpha = \lambda$ - however, on that boundary, whether we write the profit in terms of p_a or p_b, we obtain the same profit because on that particular line, $p_a = p_b$.

Bringing all the terms to a common denominator, we can write $\Psi_b(\alpha, \lambda)\,|_{\alpha=\lambda} = \frac{q_7}{q_8}$, where:

$$\begin{aligned} q_7 =& -\lambda^4 - \lambda^3 - 4\lambda p_b^3 + 4p_b^3 + 4\lambda^2 p_b^2 - 12\lambda p_b^2 - \lambda^4 p_b + 9\lambda^2 p_b, \\ q_8 =& \, 4\lambda(\lambda + \lambda p_b - p_b) > 0. \end{aligned}$$

Therefore, the sign of $\Psi_b(\alpha, \lambda)\,|_{\alpha=\lambda}$ is the same as the sign of the numerator, q_7.

We use $G_{SUB,b}(p_b) = 0$ to reduce the expression of q_7 from a cubic polynomial in p_b to a quadratic one, as follows:

$$q_7 = \frac{\lambda^2\left(\lambda^3 + 3\lambda + 2(1-\lambda)p^2 - (3-\lambda)(\lambda+1)^2 p\right)}{1-\lambda}.$$

Denote:

$$\begin{aligned} R &\triangleq 2(1-\lambda), \\ S &\triangleq -(3-\lambda)(\lambda+1)^2, \\ T &\triangleq \lambda^3 + 3\lambda. \end{aligned}$$

Then $\frac{(1-\lambda)q_7}{\lambda^2} = Rp_b^2 + Sp_b + T$. Define quadratic function $\hat{H}_{SUB,PL}(p) \triangleq Rp^2 + Sp + T$. In this range of the parameter space, it can be shown that $S^2 - 4RT > 0$. Hence, there are two real solutions to the equation $\hat{H}_{SUB,PL}(p) = 0$, namely:

$$p_{\hat{H}1} = \frac{-S - \sqrt{S^2 - 4RT}}{2R} \quad \text{and} \quad p_{\hat{H}2} = \frac{-S + \sqrt{S^2 - 4RT}}{2R}.$$

It can be shown that $\frac{\lambda}{2} < p_{\hat{H}1} < \lambda < p_{\hat{H}2}$. Recall that p_b is the unique solution of $G_{SUB,b}(p) = 0$. Moreover, from the proof of Prop. C.2, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(p_{\hat{H}1}) < 0 = G_{SUB,b}(p_b)$. Hence, $p_b < p_{\hat{H}1} < p_{\hat{H}2}$. Furthermore, $R > 0$, which indicates that $\hat{H}_{SUB,PL}(p)$ is convex. Therefore, $\hat{H}_{SUB,PL}(p_b) > 0$. Hence, in this region of the parameter space:

$$\Psi_b(\alpha, \lambda)\Big|_{\alpha=\lambda} > 0.$$

Thus, $\Psi_b(\alpha, \lambda)\Big|_{\alpha=\alpha^\dagger(\lambda)} < 0$ and $\Psi_b(\alpha, \lambda)\Big|_{\alpha=\lambda} > 0$. Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_b(\lambda)}$, which separates the optimality regions for *CE-SUB* and *CE-PL*, and which falls between boundaries $\alpha = \alpha^\dagger$ and $\alpha = \lambda$ (with the 3 lines converging when $\lambda \to 1$). It satisfies:

$$p_b\left(1 - \frac{p_b}{\lambda} + 1 - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha_b(\lambda)}}\right) = \frac{\alpha_b(\lambda)(1+\lambda)}{4}.$$

Since existence and uniqueness are satisfied, $\alpha_b(\lambda)$ is properly defined as a function. Moreover, since $\Psi_b(\alpha_b(\lambda), \lambda) = 0$, by differentiation w.r.t. λ, we obtain $\frac{\partial \alpha_b(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \alpha}} > 0$. Hence, $\alpha_b(\lambda)$ is increasing in λ.

As $\alpha = 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$ is decreasing in λ, there exists a unique intersection point between $\alpha = 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$ and $\alpha_b(\lambda)$. Defining this point as $\{\lambda_{2,b}, \alpha_b(\lambda_{2,b})\}$, we can immediately see that $\{\lambda_2, \alpha_a(\lambda_2)\}$ and $\{\lambda_{2,b}, \alpha_b(\lambda_{2,b})\}$ satisfy exactly the same conditions. Due to the uniqueness properties discussed above, we have $\lambda_{2,b} = \lambda_2$ and $\alpha_a(\lambda_2) = \alpha_b(\lambda_2)$.

Thus, $\alpha_b(\lambda)$ is properly defined and increasing on $[\lambda_2, 1)$.

Moreover, we extend the domain of α_b to include 1. We define asymptotically $\alpha_b(1) = 1 = lim_{\lambda \uparrow 1} \alpha_b(\lambda)$.

- **Definition of λ_t and $\alpha_c(\lambda)$. Monotonicity of $\alpha_c(\lambda)$.**
  Next, we compare S and TLF in the region $0 < \alpha < \alpha^\ddagger$. Denote the profit difference between S and TLF strategies in this region as:

$$\Psi_c(\alpha, \lambda) \triangleq \frac{1}{16(1-\alpha)} - \frac{\lambda}{4}.$$

Then:

$$\frac{\partial \Psi_c(\alpha, \lambda)}{\partial \alpha} > 0 > \frac{\partial \Psi_c(\alpha, \lambda)}{\partial \lambda}.$$

Therefore, a threshold (crossing) boundary between optimality regions for S and TLF within this particular region of the parameter space ($0 < \alpha < \alpha^\ddagger$) is unique for every λ and for every α (i.e., if we look vertically or horizontally), *if it exists.*

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\lambda = 0$ and $\lambda = 1$ and examine the sign of $\Psi_c(\alpha, \lambda)$ along these boundaries.

  - On the boundary $\lambda = 0$, we obtain:

$$\Psi_c(\alpha, \lambda)\Big|_{\lambda=0} = \frac{1}{16(1-\alpha)} > 0.$$

  - On the boundary $\lambda = 1$, as $\alpha < \alpha^\ddagger < \frac{3}{4}$, we obtain:

$$\Psi_c(\alpha, \lambda)\Big|_{\lambda=1} = \frac{1}{16(1-\alpha)} - \frac{1}{4} < 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_c(\lambda)}$, which separates the optimality regions for S and TLF. It satisfies:

$$\frac{1}{16(1-\alpha_c(\lambda))} - \frac{\lambda}{4} = 0.$$

We obtain: $\alpha_c(\lambda) \triangleq 1 - \frac{1}{4\lambda}$. Also, $\alpha_c(\lambda)$ is increasing in λ. Let λ_t be the solution to $\alpha_c(\lambda) = 0$. It immediately follows that $\lambda_t = \frac{1}{4}$.

As $\alpha^{\ddagger}(\lambda)$ is decreasing in λ, there exists a unique intersection point between $\alpha^{\ddagger}(\lambda)$ and $\alpha_c(\lambda)$. Defining this intersection point as $\{\lambda_c, \alpha_c(\lambda_c)\}$, we can numerically get that $\lambda_c \approx 0.2789$. Then, $\alpha_c(\lambda)$ is properly defined and increasing on $\lambda \in [\frac{1}{4}, \lambda_c]$.

- **Definition of $\alpha_d(\lambda)$ and λ_3. Monotonicity of $\alpha_d(\lambda)$.**
  We further compare π^*_{CE-SUB} under the first case in Prop. C.2, i.e., $0 < \alpha \leq \lambda \leq 1$, and π^*_{TLF}.

  We focus first on the case $\lambda < 1$. In this region, define the difference between optimal profits under *CE-SUB* and *TLF* as:

  $$\Psi_d(\alpha, \lambda) = p_a \left(2 - \frac{p_a}{\alpha} - \frac{p_a}{1 + p_a - \frac{p_a}{\alpha}}\right) - \frac{\lambda}{4}.$$

  Let's next try to understand the monotonicity of $\Psi_d(\alpha, \lambda)$ with respect to α and λ. After taking derivatives and applying the Envelope theorem with respect to π^*_{CE-SUB}, given that $p_a(\alpha, \lambda)$ represents the maximizing price for *CE-SUB* in this region, we obtain:

  $$\frac{\partial \Psi_d(\alpha, \lambda)}{\partial \alpha} = p_a^2 \left(\frac{1}{\alpha^2} + \frac{p_a}{(\alpha - (1-\alpha)p_a)^2}\right) > 0,$$
  $$\frac{\partial \Psi_d(\alpha, \lambda)}{\partial \lambda} = -\frac{1}{4} < 0.$$

  Therefore, for each λ (α) there can be *at most one* crossing point that separates the optimality regions for *CE-SUB* and *TLF* in this region as we move α (λ). Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space ($0 < \alpha \leq \lambda$).

  We look at two particular delimiting boundaries for this region, namely $\alpha \to 0$ and $\alpha = \lambda$ and examine the sign of $\Psi_d(\alpha, \lambda)$ along these boundaries.

  - On the boundary $\alpha \to 0$, under *CE-SUB*, the firm can only jump start adoption through a subscription rate $p_a \to 0$. Thus, $\lim_{\alpha \downarrow 0} \pi^*_{CE-SUB} = 0$. Hence:

    $$\lim_{\alpha \downarrow 0} \Psi_d(\alpha, \lambda) = 0 - \frac{\lambda}{4} \leq 0.$$

  - On the boundary $\alpha = \lambda$, we obtain:

    $$\Psi_d(\alpha, \lambda)\Big|_{\alpha=\lambda} = \frac{4(1-\lambda)p_a^3 - 4(3-\lambda)\lambda p_a^2 + (9-\lambda)\lambda^2 p_a - \lambda^3}{4\lambda(\lambda + \lambda p_a - p_a)}.$$

Bringing all the terms to a common denominator, we can write $\Psi_d(\alpha,\lambda)\Big|_{\alpha=\lambda} = \frac{q_9}{q_{10}}$, where:

$$\begin{aligned} q_9 &= 4(1-\lambda)p_a^3 - 4(3-\lambda)\lambda p_a^2 + (9-\lambda)\lambda^2 p_a - \lambda^3, \\ q_{10} &= 4\lambda(\lambda + \lambda p_a - p_a) > 0. \end{aligned}$$

The second inequality holds because $p_a \in \left(\frac{\alpha}{2}, \alpha\right)$ and, in this region, $\alpha < \lambda$. Therefore, the sign of $\Psi_d(\alpha,\lambda)\Big|_{\alpha=\lambda}$ is the same as the sign of the numerator, q_9.

We use $G_{SUB,a}(p_a) = 0$ to reduce the expression of q_9 from a cubic polynomial in p_a to a quadratic one, as follows:

$$q_9 = \frac{\lambda^2\left(\lambda(\lambda+3) + 2(1-\lambda)p_a^2 + (-3+(\lambda-6)\lambda)p_a\right)}{1-\lambda}.$$

Denote:

$$\begin{aligned} U &\triangleq 2(1-\lambda), \\ V &\triangleq -3 + (\lambda-6)\lambda, \\ W &\triangleq \lambda(\lambda+3). \end{aligned}$$

Then $\frac{(1-\lambda)q_9}{\lambda^2} = Up_a^2 + Vp_a + W$. Define quadratic function $H_{SUB,TLF}(p) \triangleq Up^2 + Vp + W$. In this range of the parameter space, it can be shown that $V^2 - 4UW > 0$. Hence, there are two real solutions to the equation $H_{SUB,TLF}(p) = 0$, namely:

$$\tilde{\tilde{p}}_{H1} = \frac{-V - \sqrt{V^2 - 4UW}}{2U} \quad \text{and} \quad \tilde{\tilde{p}}_{H2} = \frac{-V + \sqrt{V^2 - 4UW}}{2U}.$$

It can be shown that $\frac{\alpha}{2} < \tilde{\tilde{p}}_{H1} < \alpha < \tilde{\tilde{p}}_{H2}$. Recall that p_a is the unique solution of $G_{SUB,a}(p) = 0$. Moreover, from the proof of Prop. C.2, we know that $G_{SUB,a}(p) > 0$ on $(-\infty, p_a)$ and $G_{SUB,a}(p) < 0$ on (p_a, ∞). It can be proved directly that $G_{SUB,a}(\tilde{\tilde{p}}_{H1}) < 0 = G_{SUB,a}(p_a)$. Hence, $p_a < \tilde{\tilde{p}}_{H1} < \tilde{\tilde{p}}_{H2}$. Furthermore, $U > 0$, which indicates that $H_{SUB,TLF}(p)$ is convex. Therefore, $H_{SUB,TLF}(p_a) > 0$. Hence, in this region of the parameter space:

$$\Psi_d(\alpha,\lambda)\Big|_{\alpha=\lambda} > 0.$$

Thus, $\Psi_d(\alpha,\lambda)\Big|_{\alpha\to 0} < 0$ and $\Psi_d(\alpha,\lambda)\Big|_{\alpha=\lambda} > 0$. Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_d(\lambda)}$, which separates the optimality regions for *CE-SUB* and *TLF* (i.e., and $\Psi_d(\alpha_d(\lambda),\lambda) = 0$), which falls between boundaries $\alpha = 0$ and $\alpha = \lambda$. It satisfies:

$$p_a\left(2 - \frac{p_a}{\alpha_d(\lambda)} - \frac{p_a}{1 + p_a - \frac{p_a}{\alpha_d(\lambda)}}\right) = \frac{\lambda}{4}.$$

Since existence and uniqueness are satisfied, $\alpha_d(\lambda)$ is properly defined as a function. In terms of the domain of $\alpha_d(\lambda)$, given that $\Psi_d(\alpha,\lambda)\Big|_{\alpha\to 0} = 0 - \frac{\lambda}{4} \le 0$, the intersection of $\alpha_d(\lambda)$ and $\alpha = 0$ (x-axis) line can only happen when $\lambda = 0$. Hence $\alpha_d(\lambda)$ is well defined on $(0,1)$ domain. Moreover, since $\Psi_d(\alpha_d(\lambda),\lambda) = 0$, by differentiation w.r.t. λ, we obtain $\frac{\partial \alpha_d(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \alpha}} > 0$. Hence, $\alpha_d(\lambda)$ is increasing in λ.

For the case $\lambda = 1$, we have $\alpha_d(1) = \lim_{\lambda\to 1} \alpha_d(\lambda) = \tilde{\alpha}$, where $\tilde{\alpha}$ was defined in Proposition 1.

Next we check whether $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$ have a crossing point. First, let's check that α_c and α_d are defined in overlapping regions. $\alpha_c(\lambda)$ is defined on $\lambda \in \left[\frac{1}{4}, \lambda_c\right]$, where $\lambda_C \approx 0.2789$. It is easy to check that $\alpha_c(\lambda) = 1 - \frac{1}{4\lambda} < \lambda$. Thus, any point $\{\lambda, \alpha_c(\lambda)\}$ with $\lambda \in \left[\frac{1}{4}, \lambda_c\right]$ falls inside the bigger region $0 < \alpha \le \lambda \le 1$, which is also the region where $\alpha_d(\lambda)$ is defined.

Both $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$ are increasing in λ, as previously proved. In this region (i.e., $\lambda \in \left[\frac{1}{4}, \lambda_c\right]$, with $\lambda_C \approx 0.2789$), using $\alpha_d(\lambda) < \lambda$ and $p_a\left(\alpha_d(\lambda), \lambda\right) \in \left(\frac{\alpha_d(\lambda)}{2}, \alpha_d(\lambda)\right)$, it can be shown that:

$$\frac{\partial \alpha_d(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \alpha}}\Bigg|_{\alpha=\alpha_d(\lambda)} = \frac{1}{4p_a^2\left(\frac{1}{\alpha_d(\lambda)^2} + \frac{p_a}{(\alpha_d(\lambda)-(1-\alpha_d(\lambda))p_a)^2}\right)} < \frac{1}{4\lambda^2} = \frac{\partial \alpha_c(\lambda)}{\partial \lambda}.$$

Therefore, there can be *at most one* intersection point between $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$ in this region.

Given that α_d is defined on $(0,1]$ and α_c is defined on $\left[\frac{1}{4}, \lambda_c\right]$, with $(\lambda_c, \alpha_c(\lambda_c))$ being on $\alpha^{\ddagger}$ line, for $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$ to intersect, it is sufficient to show that $\alpha_d\left(\frac{1}{4}\right) \ge \alpha_c\left(\frac{1}{4}\right)$ and $\alpha_d(\lambda_c) \le \alpha_c(\lambda_c)$. Since α_d is increasing, it can be immediately seen that:

$$\alpha_d\left(\frac{1}{4}\right) \ge 0 = \alpha_c\left(\frac{1}{4}\right).$$

Moreover, through numerical derivation, it can be shown that:

$$\alpha_d(\lambda) - \alpha_c(\lambda)\Big|_{\lambda=\lambda_c} \approx 0.0882 - 0.1036 < 0.$$

Therefore, there exists one unique intersection point between $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$, which we define as $\{\lambda_3, \alpha_c(\lambda_3)\}$. Then, we have:

$$\pi^*_{CE-SUB}(\lambda_3, \alpha_c(\lambda_3)) = \pi^*_S(\lambda_3, \alpha_c(\lambda_3)) = \pi^*_{TLF}(\lambda_3, \alpha_c(\lambda_3)).$$

More precisely, λ_3 satisfies:

$$\lambda_3 \in \left[\frac{1}{4}, \lambda_c\right] \quad \text{and} \quad \frac{1}{16(1-\alpha_c(\lambda_3))} = p_a\left(2 - \frac{p_a}{\alpha_c(\lambda_3)} - \frac{p_a}{1 + p_a - \frac{p_a}{\alpha_c(\lambda_3)}}\right) = \frac{\lambda_3}{4}.$$

Also, we can numerically get $\lambda_3 \approx 0.272 < \lambda_c$. $\{\lambda_3, \alpha_c(\lambda_3)\}$ falls into the region $0 < \alpha < \alpha^{\ddagger}$.

Since $\alpha_d(\lambda)$ is properly defined on $(0, 1]$, obviously it is also properly defined on $[\lambda_3, 1]$.

- **Definition of threshold constants α_t.**
We further compare π^*_{CE-SUB} under the first case in Prop. C.2 and π^*_S under the first case in Prop C.4. Specifically, we look at the parameter region at the intersection of constraints $0 < \alpha < \alpha^\ddagger$ and $0 < \alpha \leq \lambda$. In this region, define the difference between optimal profits under *CE-SUB* and *S* as:

$$\Psi_t(\alpha, \lambda) \triangleq p_a \left(2 - \frac{p_a}{\alpha} - \frac{p_a}{1 + p_a - \frac{p_a}{\alpha}}\right) - \frac{1}{16(1-\alpha)}.$$

Since p_a is the unique solution of $G_{SUB,a}(p_a) = 0$, p_a does not depend on λ. Therefore, $\Psi_t(\alpha, \lambda)$ does not depend on λ. After taking derivatives and applying the Envelope theorem with respect to π^*_{CE-SUB}, given that $p_a(\alpha, \lambda)$ represents the maximizing price for *CE-SUB* in this region, we obtain:

$$\frac{\partial \Psi_t(\alpha, \lambda)}{\partial \alpha} = p_a^2 \left(\frac{1}{\alpha^2} + \frac{p_a}{(\alpha - (1-\alpha)p_a)^2}\right) - \frac{1}{16(1-\alpha)^2},$$
$$\frac{\partial \Psi_t(\alpha, \lambda)}{\partial \lambda} = 0.$$

Let's check the sign of $\frac{\partial \Psi_t(\alpha,\lambda)}{\partial \alpha}$. Bringing all the terms to a common denominator, we can write $\frac{\partial \Psi_t(\alpha,\lambda)}{\partial \alpha} = \frac{q_{11}}{q_{12}}$, where:

$$q_{11} = 16(1-\alpha)^4 p_a^4 + 16\alpha(3\alpha - 2)(1-\alpha)^2 p_a^3 + 15\alpha^2(1-\alpha)^2 p_a^2 + 2\alpha^3(1-\alpha)p_a - \alpha^4,$$
$$q_{12} = 16(1-\alpha)^2\alpha^2(\alpha + \alpha p_a - p_a)^2 > 0.$$

Therefore, the sign of $\frac{\partial \Psi_t(\alpha,\lambda)}{\partial \alpha}$ is the same as the sign of the numerator, q_{11}.

Recall from Prop. C.2 that p_a is the unique solution to the cubic equation $G_{SUB,a}(p_a) = 0$. We use this property of p_a to reduce the expression of q_{11} from a quartic polynomial in p_a to a quadratic one, as follows:

$$q_{11} = \left(4\alpha^4 - 16\alpha^3 - 5\alpha^2 + 2\alpha + 15\right)\alpha^2 p_a^2 + 2\left(4\alpha^3 - 8\alpha^2 + 3\alpha - 15\right)\alpha^3 p_a + \left(8\alpha^2 - 8\alpha + 15\right)\alpha^4.$$

Denote:

$$X \triangleq \alpha^2\left(4\alpha^4 - 16\alpha^3 - 5\alpha^2 + 2\alpha + 15\right),$$
$$Y \triangleq 2\alpha^3\left(4\alpha^3 - 8\alpha^2 + 3\alpha - 15\right),$$
$$Z \triangleq \alpha^4\left(8\alpha^2 - 8\alpha + 15\right).$$

Then $q_{11} = Xp_a^2 + Yp_a + Z$. Define quadratic function $H_{SUB,S}(p) \triangleq Xp^2 + Yp + Z$. In this range of the parameter space, it can be shown that:

$$Y^2 - 4XZ = 16\alpha^8(\alpha(\alpha(-4(\alpha - 6)\alpha - 15) + 4) + 55) > 0.$$

Hence, there are two real solutions of $H_{SUB,S}(p) = 0$, namely:

$$\bar{\bar{p}}_{H1} = \frac{-Y - \sqrt{Y^2 - 4XZ}}{2X} \quad \text{and} \quad \bar{\bar{p}}_{H2} = \frac{-Y + \sqrt{Y^2 - 4XZ}}{2X}.$$

It can be shown that $\bar{\bar{p}}_{H1} < \bar{\bar{p}}_{H2} < \frac{\alpha}{1-\alpha}$. From the proof of Prop. C.2, we know that $G_{SUB,a}(p) > 0$ on $(-\infty, p_a)$ and $G_{SUB,a}(p) < 0$ on (p_a, ∞). It can be proved directly that $G_{SUB,a}(\bar{\bar{p}}_{H1}) < 0$. Hence, $p_a < \bar{\bar{p}}_{H1} < \bar{\bar{p}}_{H2}$.

Furthermore, it can be shown that $X > 0$, which indicates that $H_{SUB,S}(p)$ is convex. Therefore, $H_{SUB,PL}(p_a) > 0$. Thus, $\frac{\partial \Psi_t(\alpha,\lambda)}{\partial \alpha} > 0$. Thus, for any given λ, when we increase α, there can be *at most one* crossing point that separates the optimality regions for *CE-SUB* and *S*, and, moreover, the crossing (if it exists) can be only from *S* to *CE-SUB* as α increases. Such a separating threshold line, if it exists, has to be horizontal (i.e., constant for any λ for which it exists in this region) since $\Psi_t(\alpha, \lambda)$ is independent of λ (because p_a is independent of λ).

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space ($0 < \alpha < \min\{\lambda, \alpha^\ddagger\}$). Since we established that such a threshold will be a horizontal line cutting through this region, it is enough to show that it exists at a particular λ. Consider $\{\lambda_4, \alpha_4\}$, with $\lambda_4 = \alpha_4$, to be the unique intersection between lines $\alpha = \lambda$ and $\alpha = \alpha^\ddagger$.[C-3] Numerical analysis reveals that $\lambda_4 = \alpha_4 \approx 0.1195$. We examine the sign of $\Psi_t(\alpha, \lambda)$ at boundary points $\{\lambda_4, 0\}$ and $\{\lambda_4, \alpha_4\}$:

$$\Psi_t(\alpha, \lambda)\Big|_{\lambda=\lambda_4,\, \alpha\downarrow 0} = 0 - \frac{1}{16} < 0,$$
$$\Psi_t(\alpha, \lambda)\Big|_{\lambda=\alpha=\lambda_4} \approx 0.0909 - 0.0710 > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary which separates the optimality regions for *S* and *CE-SUB* and which does not change with λ. And it is straight forward that the boundary line goes through the point $\{\lambda_3, \alpha_c(\lambda_3)\}$ since it is the point when $\pi^*_S = \pi^*_{CE-SUB}$. We define this threshold as constant $\underline{\alpha_t \triangleq \alpha_c(\lambda_3)}$. This horizontal boundary extends from $\{\alpha_t, \alpha_t\}$ to $\{\lambda_3, \alpha_t\}$.

- **Definition and monotonicity of $\lambda_x(\alpha)$.**
  Finally, we compare π^*_{CE-SUB} under the second case in Prop. C.2 and π^*_S under the first case in Prop. C.4. More specifically, we explore the parameter space at the intersection of constraints $\lambda < \alpha \leq \alpha^\dagger$ and $0 < \alpha < \alpha^\ddagger$. We denote the difference between optimal profits under strategies *CE-SUB* and *S* as:

$$\Psi_x(\alpha, \lambda) \triangleq p_b \left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha}}\right) - \frac{1}{16(1-\alpha)}.$$

[C-3] we know that the intersection is unique because $\alpha = \lambda$ is increasing in λ, whereas $\alpha^\ddagger$ is decreasing in λ.

As p_b maximizes π^*_{CE-SUB}, using Envelope theorem, we get:

$$\begin{aligned}\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \alpha} &= \frac{p_b^3}{(\alpha-(1-\alpha)p_b)^2} - \frac{1}{16(1-\alpha)^2}\\ \frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda} &= \frac{p_b^2}{\lambda^2} > 0.\end{aligned}$$

As it turns out, in this range of the parameter space, $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \alpha}$ changes signs. As such, it is not possible to characterize the threshold between S and *CE-SUB* as a function of λ (there exist values of λ for which increasing α leads to multiple crossings between optimality regions for S and *CE-SUB*).

Nevertheless, moving horizontally, given that $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda} > 0$, a threshold (crossing) boundary between optimality regions for *CE-SUB* and S, within this particular region of the parameter space, is unique for every α, *if it exists.*

Next, we show that such a threshold boundary *does* indeed exist in this region.

We look at two particular cases for this region:

- First, we consider points on the boundary $\alpha = \alpha^\dagger(\lambda)$. Note that, in this region, we have $\lambda < \alpha < \alpha^\ddagger(\lambda)$ (since we consider the intersection of constraints $\lambda < \alpha \leq \alpha^\dagger$ and $0 < \alpha < \alpha^\ddagger$). Given that $\alpha^\ddagger$ is decreasing, as shown above, it means that in this region we have $\lambda < \alpha^\ddagger(\lambda) \leq \alpha^\ddagger(0) \approx 0.1352 < \frac{1}{3}$.[C-4] From Prop. C.2, given that in this parameter region we have $\lambda < \frac{1}{3}$, we consequently get $\alpha^\dagger = \tilde{\alpha}^\dagger$ (see equation (C.1)), which satisfies $\Xi(\alpha,\lambda) = 0$ and $\frac{\partial \Xi(\alpha,\lambda)}{\partial \alpha} < 0$. Similarly, from the Envelope theorem, we get:

  $$\frac{\partial \Xi(\alpha,\lambda)}{\partial \lambda} = \frac{p_b^2}{\lambda^2} > 0.$$

  Therefore,

  $$\frac{\partial \alpha^\dagger(\lambda)}{\partial \lambda} = -\left.\frac{\frac{\partial \Xi(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Xi(\alpha,\lambda)}{\partial \alpha}}\right|_{\alpha=\alpha^\dagger(\lambda)} > 0.$$

  Thus, $\alpha^\dagger(\lambda) = \tilde{\alpha}^\dagger(\lambda)$ is strictly increasing in λ in this region. Therefore, it is invertible. On the boundary $\alpha = \alpha^\dagger(\lambda)$, we obtain.

  $$\left.\Psi_x(\alpha,\lambda)\right|_{\lambda=\alpha^{\dagger-1}(\alpha)} = \frac{\alpha}{(\sqrt{\alpha}+1)^2} - \frac{1}{16(1-\alpha)} < 0. \tag{C.5}$$

- Next, we consider the intersection point between $\alpha = \alpha^\ddagger$ and $\alpha = \lambda$, which is $\{0.1195, 0.1195\}$.

[C-4] We can achieve the same conclusion the following way. The intersection point between boundaries $\alpha = \alpha^\ddagger(\lambda)$ (decreasing) and $\alpha = \lambda$ (increasing), which occurs approximately at $\{0.1195, 0.1195\}$, achieves the maximum λ for this region, which is smaller than $\frac{1}{3}$.

At this point, we obtain:

$$\Psi_x(\alpha,\lambda)\Big|_{\alpha=\lambda=0.1195} = 0.0909 - 0.0710 > 0. \tag{C.6}$$

Therefore, in this parameter region, there exists a sub-region where $\Psi_x < 0$ (S dominates *CE-SUB*) and a sub-region where $\Psi_x > 0$ (*CE-SUB* dominates S). Given that, for any λ, as we increase α, there can be at most one crossing point between optimality regions for S and *CE-SUB*, then there exists a unique threshold boundary, which we define as $\underline{\lambda_x(\alpha)}$, which separates the optimality regions for *CE-SUB* and S. It satisfies:

$$\frac{1}{16(1-\alpha)} = p_b\left(2 - \frac{p_b}{\lambda_x(\alpha)} - \frac{p_b}{1+p_b-\frac{p_b}{\alpha}}\right).$$

Let's next examine the domain of $\lambda_x(\alpha)$. For that purpose, we look at the monotonicity of $\Psi_x(\alpha,\lambda)$ in terms of λ on two particular boundaries:

- First, we consider the line $\alpha = \lambda$ (boundary in limit). On this line, we get:

$$\Psi_x(\alpha,\lambda)\Big|_{\lambda=\alpha} = p_b\left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1+p_b-\frac{p_b}{\lambda}}\right) - \frac{1}{16(1-\lambda)}.$$

  Again, we reminder the reader that we could have written the profit for *CE-SUB* in terms of p_a at the boundary when $\alpha = \lambda$ - however, on this boundary, whether we write the profit in terms of p_a or p_b, we obtain the same profit because on that particular line, $p_a = p_b$. Then,

$$\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda}\Big|_{\lambda=\alpha} = p_b^2\left(\frac{1}{\lambda^2} + \frac{p_b}{(\lambda+(\lambda-1)p_b)^2}\right) - \frac{1}{16(\lambda-1)^2}.$$

  Let's check the sign of $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda}\Big|_{\lambda=\alpha}$. Bringing all the terms to a common denominator, we can write $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda} = \frac{q_{13}}{q_{14}}$, where:

$$\begin{aligned} q_{13} =& 16(1-\lambda)^4 p_b^4 + 16\lambda(3\lambda-2)(1-\lambda)^2 p_b^3 + 15\lambda^2(1-\lambda)^2 p_b^2 + 2\lambda^3(1-\lambda)p_b - \lambda^4, \\ q_{14} =& 16(1-\lambda)^2\lambda^2(\lambda+\lambda p_b - p_b)^2 > 0. \end{aligned}$$

  Therefore, the sign of $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda}$ is the same as the sign of the numerator, q_{13}.

  Recall from Prop. C.2 that p_b is the unique solution to the cubic equation $G_{SUB,b}(p_b) = 0$. We use this property of p_b to reduce the expression of q_{13} from a quartic polynomial in p_b to a quadratic one, as follows:

$$q_{13} = \left(4\lambda^4 - 16\lambda^3 - 5\lambda^2 + 2\lambda + 15\right)p_b^2 + 2\lambda\left(4\lambda^3 - 8\lambda^2 + 3\lambda - 15\right)p_b + \lambda^2\left(8\lambda^2 - 8\lambda + 15\right).$$

Denote:

$$A_{x,1} \triangleq \left(4\lambda^4 - 16\lambda^3 - 5\lambda^2 + 2\lambda + 15\right),$$
$$B_{x,1} \triangleq 2\lambda\left(4\lambda^3 - 8\lambda^2 + 3\lambda - 15\right),$$
$$C_{x,1} \triangleq \lambda^2\left(8\lambda^2 - 8\lambda + 15\right).$$

Then $q_{13} = A_{x,1}p_b^2 + B_{x,1}p_b + C_{x,1}$. Define quadratic function $H^{\Diamond}_{SUB,S}(p) \triangleq A_{x,1}p^2 + B_{x,1}p + C_{x,1}$. In this range of the parameter space, it can be shown that:

$$B_{x,1}^2 - 4A_{x,1}C_{x,1} = 16\lambda^4(\lambda(\lambda(-4(\lambda-6)\lambda - 15) + 4) + 55) > 0.$$

Hence, there are two real solutions to the equation $H^{\Diamond}_{SUB,S}(p) = 0$, namely:

$$p^{\Diamond}_{H1} = \frac{-B_{x,1} - \sqrt{B_{x,1}^2 - 4A_{x,1}C_{x,1}}}{2A_{x,1}} \quad \text{and} \quad p^{\Diamond}_{H2} = \frac{-B_{x,1} + \sqrt{B_{x,1}^2 - 4A_{x,1}C_{x,1}}}{2A_{x,1}}.$$

It can be shown that $p^{\Diamond}_{H1} < p^{\Diamond}_{H2} < \frac{\alpha}{1-\alpha}$ when $\lambda < \frac{1}{3}$ (which is satisfied in this region of the parameter space, as per the above argument). From the proof of Prop. C.2, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(p^{\Diamond}_{H1}) < 0$. Hence, $p_b < p^{\Diamond}_{H1} < p^{\Diamond}_{H2}$.

Furthermore, it can be shown that $A_{x,1} > 0$, which indicates that $H^{\Diamond}_{SUB,S}(p)$ is convex. Therefore, $H^{\Diamond}_{SUB,S}(p_b) > 0$. Thus, $\left.\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda}\right|_{\lambda=\alpha} > 0$. Thus, on the portion of boundary $\alpha = \lambda$ within this particular region ($0 < \alpha < \alpha^{\ddagger}$), as we increase λ (or, equivalently, as we increase α), there can be *at most one* crossing point that separates the optimality regions for S and *CE-SUB*, and, moreover, the crossing (if it exists) can be only from S to *CE-SUB* as λ increases.

On the asymptotic boundary $\lambda = \alpha$, when $\alpha \to 0$, $\pi^*_S > \pi^*_{CE-SUB}$ (as per inequality (C.5), given that $\lim_{\lambda \downarrow 0} \alpha^{\dagger} = 0$); when $\alpha \to \alpha^{\ddagger}$, $\pi^*_S < \pi^*_{CE-SUB}$ (as per inequality (C.6)). Therefore, there exists a unique intersection point between $\lambda_x(\alpha)$ and $\lambda = \alpha$ within this region. And it is straight forward to see that the intersection point is $\{\alpha_t, \alpha_t\}$ since it is the point when $\pi^*_S = \pi^*_{CE-SUB}$. More precisely, when $\lambda_x(\alpha_t) = \alpha_t$.

– Second, we consider the boundary $\alpha = \alpha^{\ddagger}(\lambda)$ (boundary in limit). On this line, we can get:

$$\Psi_x(\alpha^{\ddagger}(\lambda), \lambda) = p_b\left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha^{\ddagger}(\lambda)}}\right) - \frac{1}{16(1-\alpha^{\ddagger}(\lambda))}.$$

Differentiating with respect to λ, and using Envelope theorem as p_b is maximizing π_{CE-SUB}, we obtain:

$$\frac{\partial \Psi_x}{\partial \lambda} = \left(\frac{p_b^3}{(\alpha^{\ddagger} - (1-\alpha^{\ddagger})p_b)^2} - \frac{1}{16(1-\alpha^{\ddagger})^2} \right) \frac{\partial \alpha^{\ddagger}}{\partial \lambda} + \frac{p_b^2}{\lambda^2},$$

where $\frac{\partial \alpha^{\ddagger}(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \alpha}}$, with $\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \alpha}$ and $\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \lambda}$ expressions derived in equations (C.3) and (C.4).

Let's check the sign of $\frac{\partial \Psi_x(\alpha^{\ddagger}(\lambda),\lambda)}{\partial \lambda}$. Bringing all the terms to a common denominator, we can write $\frac{\partial \Psi_x(\alpha^{\ddagger}(\lambda),\lambda)}{\partial \lambda} = \frac{q_{15}}{q_{16}}$, where,

$$\begin{aligned} q_{16} = &-(1-\alpha)\lambda^2(\alpha + \alpha p_b - p_b)^2 \\ &\times \left(16\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3) + \left(-\alpha(8\lambda+7)^2 - 16\lambda - 15\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right) > 0, \end{aligned}$$

and q_{15}, via degree reduction (since, as per Prop. C.2, p_b is the unique solution to the cubic equation $G_{SUB,b}(p_b) = 0$), can be simplified from a quartic polynomial in p_b to a quadratic function $q_{15} = A_{x,2}p_b^2 + B_{x,2}p_b + C_{x,2}$ with:

$$\begin{aligned} A_{x,2} \triangleq &\frac{1}{4}(1-\alpha)\lambda^2 \left(4\alpha(\lambda+1)\left(\alpha^3\left(48\lambda^2 - 104\lambda - 29\right) - 2\alpha^2\left(64\lambda^2 + 28\lambda + 51\right)\right.\right. \\ &\left. + \alpha\left(64\lambda^2 - 16\lambda - 29\right) + 48(\lambda+1)\right) + \left(\alpha^3\left(-192\lambda^2 + 320\lambda + 347\right) + \alpha^2\left(512\lambda^2 + 576\lambda + 301\right)\right. \\ &\left.\left. -4\alpha\left(64\lambda^2 + 80\lambda + 35\right) - 64\lambda - 60\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right), \\ B_{x,2} \triangleq &\frac{1}{2}\alpha\lambda\left(4\alpha(\lambda+1)\left(\alpha^3\left(48\lambda^3 + 8\lambda^2 + 15\lambda + 4\right) - 2\alpha^2\left(64\lambda^3 + 28\lambda^2 + 9\lambda - 6\right)\right.\right. \\ &\left. + \alpha\lambda\left(64\lambda^2 - 16\lambda - 29\right) + 48\lambda(\lambda+1)\right) + \left(-\left(\alpha^3\left(192\lambda^3 + 128\lambda^2 + 53\lambda + 49\right)\right)\right. \\ &\left.\left. + \alpha^2\left(512\lambda^3 + 576\lambda^2 + 189\lambda - 15\right) - 4\alpha\lambda\left(64\lambda^2 + 80\lambda + 35\right) - 4\lambda(16\lambda+15)\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right), \\ C_{x,2} \triangleq &\frac{1}{2}\alpha^2\lambda^2\left(2\alpha^3\left(96\lambda^3 + 148\lambda^2 + 59\lambda + 7\right) - \alpha^2\left(48\lambda^2\left(4\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} + 3\right)\right.\right. \\ &\left. + \lambda\left(200\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} + 6\right) + 43\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} + 128\lambda^3 - 10\right) \\ &+ \alpha\left(32\lambda^2\left(4\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} - 3\right) + 16\lambda\left(11\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} - 12\right)\right. \\ &\left.\left. + 85\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} - 96\right) + 2(16\lambda+15)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right), \end{aligned}$$

where, for simplicity of notation, we dropped the superscript and used α instead of $\alpha^{\ddagger}(\lambda)$. Since $q_{16} > 0$, the sign of $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda}$ is the same as the sign of the numerator, q_{15}.

Define quadratic function $\check{H}_{SUB,S}(p) \triangleq A_{x,2}p^2 + B_{x,2}p + C_{x,2}$. In this range of the parameter space, it can be shown that:

$$B_{x,2}^2 - 4A_{x,2}C_{x,2} > 0.$$

Hence, there are two real solutions to the equation $\check{H}_{SUB,S}(p) = 0$, namely:

$$\check{p}_{H1} = \frac{-B_{x,2} - \sqrt{B_{x,2}^2 - 4A_{x,2}C_{x,2}}}{2A_{x,2}} \quad \text{and} \quad \check{p}_{H2} = \frac{-B_{x,2} + \sqrt{B_{x,2}^2 - 4A_{x,2}C_{x,2}}}{2A_{x,2}}.$$

It can be shown that $\check{p}_{H1} < \check{p}_{H2} < \frac{\alpha}{1-\alpha}$ when $\lambda < \frac{1}{3}$ (which is satisfied in this region of the parameter space, as per the above argument). From the proof of Prop. C.2, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(\check{p}_{H1}) < 0$. Hence, $p_b < \check{p}_{H1} < \check{p}_{H2}$.

Furthermore, it can be shown that $A_{x,2} > 0$, which indicates that $\check{H}_{SUB,S}(p)$ is convex. Therefore, $\check{H}_{SUB,S}(p_b) > 0$. Thus, $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda} > 0$. Hence, on the line $\alpha = \alpha^{\ddagger}(\lambda)$, when we increase λ, there can be *at most one* crossing point that separates the optimality regions for S and *CE-SUB*, and, moreover, the crossing (if it exists) can be only from S to *CE-SUB* as λ increases.

As $\alpha^{\dagger}$ is increasing in λ (and spanning the entire interval $(0,1]$) and $\alpha^{\ddagger}$ is decreasing in λ, there exists a unique intersection point between $\alpha^{\dagger}$ and $\alpha^{\ddagger}$. Defining this point as $\{\lambda_{x,1}, \alpha^{\ddagger}(\lambda_{x,1})\}$, with $\alpha^{\ddagger}(\lambda_{x,1}) = \alpha^{\dagger}(\lambda_{x,1})$.

On the asymptotic boundary $\alpha = \alpha^{\ddagger}$, when $\alpha \to \alpha^{\ddagger}(\lambda_{x,1})$, $\pi^*_S > \pi^*_{CE-SUB}$ (as per inequality (C.5)); when $\alpha \to \lambda$, $\pi^*_S < \pi^*_{CE-SUB}$ (as per inequality (C.6)). Therefore, there exists a unique intersection point between $\lambda_x(\alpha)$ and $\alpha^{\ddagger}$. We define this point as $\{\lambda_x(\alpha_x), \alpha_x\}$. At this point, we have $\pi^*_S = \pi^*_{CE-PL} = \pi^*_{CE-SUB}$. As such, it can be easily seen that $\lambda_x(\alpha_x) = \lambda_1$.

As $\lambda_x(\alpha)$ only intersects once boundaries $\alpha = \lambda$ and $\alpha = \alpha^{\ddagger}(\lambda)$, it means that $\lambda_x(\alpha)$ is properly defined on $\alpha \in (\alpha_t, \alpha_x)$, as $\{\alpha, \lambda_x(\alpha)\}$ stays inside this region of the parameter space for all $\alpha \in (\alpha_t, \alpha_x)$.

Thus, we completely characterized lines α_1, α_2, and λ_x, (in particular, segments, $\alpha^{\ddagger}(\lambda)$, $\alpha_a(\cdot)$, $\alpha_b(\cdot)$, $\alpha_c(\cdot)$, $\alpha_d(\cdot)$, as well as constant thresholds λ_1, λ_2, λ_3, α_x), as well as threshold α_t.

Comparison of $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$:
All segments in $\alpha_1(\lambda)$ on $[0,1]$ (i.e., $\alpha^{\ddagger}$ on $[0,\lambda_1)$, α_a on $[\lambda_1,\lambda_2)$, and α_b on $[\lambda_2, 1]$ satisfy $\alpha_1(\lambda) \geq \lambda$ (with equality happening only when $\lambda = 1$). At the same time, all segments of $\alpha_2(\lambda)$ on $\left[\frac{1}{4}, 1\right]$ (i.e., α_c on $\left[\frac{1}{3}, \lambda_3\right)$ and α_d on $[\lambda_3, 1]$) satisfy $\alpha_2(\lambda) < \lambda$. Thus, we have:

$$\alpha_1(\lambda) > \alpha_2(\lambda) \quad \forall \lambda \in \left[\frac{1}{4}, 1\right].$$

Derivation of the dominating strategy in the entire region $0 < \alpha < 1$:

- When $\lambda \leq \alpha < 1$, it is easy to show, via direct comparison, that $\pi^*_{CE-PL} > \pi^*_{TLF}$. Therefore, *TLF* is suboptimal in this region. Then, by the definition of $\alpha_1(\lambda)$ and $\lambda_x(\alpha)$, and in light of the earlier analysis, we get:
  - When $\alpha_1(\lambda) \leq \alpha < 1$ and $\lambda \leq \alpha < 1$, *CE-PL* is the dominating strategy;
  - When $\lambda \leq \alpha < \alpha_1(\lambda)$, we have two subcases:

* When $\lambda \le \alpha < \alpha_1(\lambda)$ and $0 \le \lambda < \lambda_x(\alpha)$, then S is the dominant strategy;
* When $\lambda_x(\alpha) \le \lambda \le \alpha < \alpha_1(\lambda)$, then *CE-SUB* is the dominant strategy.

- When $0 < \alpha < \lambda$, we first show that *CE-PL* is always dominated:
  - When $0 < \alpha \le \frac{\lambda\left(4+5\lambda-4(1+\lambda)\sqrt{\lambda}\right)}{16+24\lambda-7\lambda^2-16\lambda^3}$ and $0 < \alpha < \lambda$, then it can be shown that $\pi^*_{TLF} = \frac{\lambda}{4} \ge \pi^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$. Hence, in this region, *CE-PL* is dominated.
  - When $\frac{\lambda\left(4+5\lambda-4(1+\lambda)\sqrt{\lambda}\right)}{16+24\lambda-7\lambda^2-16\lambda^3} < \alpha < 5 + 8\lambda - 4\sqrt{(\lambda+1)(4\lambda+1)}$ and $0 < \alpha < \lambda$, then $\pi^*_{TLF} = \frac{\lambda}{4} < \pi^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$. Therefore, in this region, *CE-PL* dominates *TLF*. Define the difference between optimal profits under *CE-SUB* and *CE-PL* as:

$$\Psi_e(\alpha,\lambda) \triangleq p_a\left(2 - \frac{p_a}{\alpha} - \frac{p_a}{1+p_a-\frac{p_a}{\alpha}}\right) - \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2},$$

Bringing all the terms to a common denominator, we can write $\Psi_e(\alpha,\lambda) = \frac{q_{17}}{q_{18}}$, where:

$$\begin{aligned} q_{17} &= (1-\alpha)^3 p_a^3 + (3-\alpha)\alpha(1-\alpha)^2 p_a^2 \\ &\quad - \alpha(1-\alpha)\left(2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)} + \alpha^2\left(-2\lambda^2-3\lambda+1\right) - \alpha(\lambda+3)\right)p_a \\ &\quad + \alpha^2\left(2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)} - \alpha^2\left(2\lambda^2+3\lambda+1\right) - \alpha(\lambda+1)\right), \\ q_{18} &= (1-\alpha)^2\alpha(\alpha+\alpha p_a - p_a) > 0, \end{aligned}$$

where $q_{18} > 0$ is due to the fact that $p_a \in \left(\frac{\alpha}{2}, \alpha\right)$. Therefore, the sign of $\Psi_e(\alpha,\lambda)$ is the same as the sign of q_{17}. Recall from Prop. C.2 that p_a is the unique solution to the cubic equation $G_{SUB,a}(p_a) = 0$. We use this property of p_a to reduce the expression of q_{17} from a cubic polynomial in p_a to a quadratic one, as follows:

$$\begin{aligned} q_{17} = &\frac{\alpha}{2} \\ &\times \Big((1-\alpha)^2\alpha p_a^2 \\ &\quad + 2(1-\alpha)\left(\alpha\lambda(\alpha(2\lambda+3)+1) - 2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}\right)p_a \\ &\quad - 2\alpha^3(\lambda(2\lambda+3)+2) + 4\alpha\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)} - 2\alpha^2\lambda\Big). \end{aligned}$$

Denote:

$$\begin{aligned} A_e &\triangleq (1-\alpha)^2\alpha, \\ B_e &\triangleq 2(1-\alpha)\left(\alpha\lambda(\alpha(2\lambda+3)+1) - 2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}\right), \\ C_e &\triangleq -2\alpha^3(\lambda(2\lambda+3)+2) + 4\alpha\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)} - 2\alpha^2\lambda. \end{aligned}$$

Then $\frac{2}{\alpha} \times q_{17} = A_e p_a^2 + B_e p_a + C_e$. Define quadratic function $\tilde{H}^{\diamondsuit}_{SUB,PL}(p) \triangleq A_e p^2 +$

$B_e p + C_e$. In this range of the parameter space, it can be shown that:

$$B_e^2 - 4A_eC_e > 0.$$

Hence, there are two real solutions to the equation $\tilde{H}^{\Diamond}_{SUB,PL}(p) = 0$, namely:

$$\tilde{p}^{\Diamond}_{H1} = \frac{-B_e - \sqrt{B_e^2 - 4A_eC_e}}{2A_e} \quad \text{and} \quad \tilde{p}^{\Diamond}_{H2} = \frac{-B_e + \sqrt{B_e^2 - 4A_eC_e}}{2A_e}.$$

It can be shown that $\tilde{p}^{\Diamond}_{H1} < \tilde{p}^{\Diamond}_{H2} < \frac{\alpha}{1-\alpha}$. From the proof of Prop. C.2, we know that $G_{SUB,a}(p) > 0$ on $(-\infty, p_a)$ and $G_{SUB,a}(p) < 0$ on (p_a, ∞). It can be proved directly that $G_{SUB,a}(\tilde{p}^{\Diamond}_{H1}) < 0$. Hence, $p_a < \tilde{p}^{\Diamond}_{H1} < \tilde{p}^{\Diamond}_{H2}$.

Furthermore, since $A_e > 0$, $\tilde{H}^{\Diamond}_{SUB,PL}(p)$ is convex. Therefore, $\tilde{H}^{\Diamond}_{SUB,PL}(p_a) > 0$. Thus, $\Psi_e(\alpha, \lambda) > 0$, meaning that, in this region, *CE-PL* is dominated by *CE-SUB*.

– When $5 + 8\lambda - 4\sqrt{(\lambda+1)(4\lambda+1)} \le \alpha < 1$, we show that *CE-PL* is dominated by *CE-SUB*. In this region, define the difference between optimal profits under *CE-SUB* and *CE-PL* as:

$$\Psi_f(\alpha, \lambda) \triangleq p_a \left(2 - \frac{p_a}{\alpha} - \frac{p_a}{1 + p_a - \frac{p_a}{\alpha}} \right) - \frac{1}{4}\alpha(\lambda + 1).$$

Bringing all the terms to a common denominator, we can write $\Psi_f(\alpha, \lambda) = \frac{q_{19}}{q_{20}}$, where:

$$\begin{aligned} q_{19} &= p_a^3(4 - 4\alpha) + 4p_a^2(\alpha - 3)\alpha + p_a\alpha^2(-\alpha(\lambda+1) + \lambda + 9) - \alpha^3(\lambda + 1), \\ q_{20} &= 4\alpha(\alpha + \alpha p_a - p_a) > 0, \end{aligned}$$

where $q_{20} > 0$ is due to the fact that $p_a \in \left(\frac{\alpha}{2}, \alpha\right)$. Therefore, the sign of $\Psi_f(\alpha, \lambda)$ is the same as that of q_{19}. Recall from Prop. C.2 that p_a is the unique solution to the cubic equation $G_{SUB,a}(p_a) = 0$. We use this property of p_a to reduce the expression of q_{19} from a cubic polynomial in p_a to a quadratic one, as follows:

$$q_{19} = A_f p_a^2 + B_f p_a + C_f,$$

with

$$\begin{aligned} A_f &\triangleq 2(1 - \alpha), \\ B_f &\triangleq \alpha(\alpha - 6 - (2 - \alpha)\lambda) + \lambda - 3, \\ C_f &\triangleq \alpha(\alpha - 1)\lambda + \alpha + 3). \end{aligned}$$

Define quadratic function $\check{H}^{\Diamond}_{SUB,PL}(p) \triangleq A_f p^2 + B_f p + C_f$. In this range of the parameter space, it can be shown that:

$$B_f^2 - 4A_fC_f = (\alpha((\alpha - 2)\lambda + \alpha - 6) + \lambda - 3)^2 - 8(1 - \alpha)\alpha((\alpha - 1)\lambda + \alpha + 3) > 0.$$

Hence, there are two real solutions to the equation $\check{H}^{\Diamond}_{SUB,PL}(p) = 0$, namely:

$$\check{p}^{\Diamond}_{H1} = \frac{-B_f - \sqrt{B_f^2 - 4A_fC_f}}{2A_f} \quad \text{and} \quad \check{p}^{\Diamond}_{H2} = \frac{-B_f + \sqrt{B_f^2 - 4A_fC_f}}{2A_f}.$$

It can be shown that $\check{p}^{\Diamond}_{H1} < \check{p}^{\Diamond}_{H2} < \frac{\alpha}{1-\alpha}$. From the proof of Prop. C.2, we know that $G_{SUB,a}(p) > 0$ on $(-\infty, p_a)$ and $G_{SUB,a}(p) < 0$ on (p_a, ∞). It can be proved directly that $G_{SUB,a}(\check{p}^{\Diamond}_{H1}) < 0$. Hence, $p_a < \check{p}^{\Diamond}_{H1} < \check{p}^{\Diamond}_{H2}$.

Furthermore, since $A_f > 0$, $\check{H}^{\Diamond}_{SUB,PL}(p)$ is convex. Therefore, $\check{H}^{\Diamond}_{SUB,PL}(p_a) > 0$. Thus, $\Psi_f(\alpha, \lambda) > 0$, meaning that, in this region, *CE-PL* is dominated by *CE-SUB*.

Since *CE-PL* is always dominated when $0 < \alpha < \lambda$, in this region we only need to compare *CE-SUB*, *TLF*, and *S*. By the definition of $\alpha_2(\lambda)$ and α_t, and in light of the earlier analysis, in the region $0 < \alpha < \lambda$ we get:

– If $\max\{\alpha, \frac{1}{4}\} < \lambda \leq 1$ and $0 < \alpha < \alpha_2(\lambda)$, then *TLF* is the dominating strategy;
– Else, if $\alpha_t \leq \alpha < \lambda$, then *CE-SUB* is the dominant strategy;
– Else, *S* is the dominant strategy.

This completes the mapping of dominant strategy to the parameter space (we discussed the case $\alpha \geq 1$ at the very beginning of the proof).

Social welfare comparison.
It can be shown with relative ease, through direct comparisons of closed form solutions, that $SW^*_{TLF} = \frac{3\lambda}{8} + \frac{1}{2} \geq \max\{SW^*_{CE-PL}, SW^*_S\}$. Thus, we only have to compare SW^*_{TLF} with $SW_{CE-SUB,a}$, $SW_{CE-SUB,b}$, and $\frac{2\sqrt{\alpha}+1}{2(\sqrt{\alpha}+1)^2}$ for $\alpha \in (0,1)$. From Prop C.2, we know that $p_a \in \left(\frac{\alpha}{2}, \alpha\right)$ and $p_b \in \left(\frac{\lambda}{2}, \lambda\right)$. It is straightforward to see that:

$$SW_{CE-SUB,a} = \frac{1}{2}\left(1 + \lambda - \frac{\lambda p_a^2}{\alpha^2} - \frac{p_a^2}{\left(1 + p_a - \frac{p_a}{\alpha}\right)^2}\right) < \frac{1}{2}\left(1 + \lambda - \frac{\lambda p_a^2}{\alpha^2}\right) < \frac{3\lambda}{8} + \frac{1}{2} = SW^*_{TLF}.$$

$$SW_{CE-SUB,b} = \frac{1}{2}\left(1 + \lambda - \frac{p_b^2}{\lambda} - \frac{1}{\left(1 + p_b - \frac{p_b}{\alpha}\right)^2}\right) < \frac{1}{2}\left(1 + \lambda - \frac{p_b^2}{\lambda}\right) < \frac{3\lambda}{8} + \frac{1}{2} = SW^*_{TLF}.$$

$$\frac{2\sqrt{\alpha}+1}{2\left(\sqrt{\alpha}+1\right)^2} < \frac{1}{2} < \frac{3\lambda}{8} + \frac{1}{2} = SW^*_{TLF}.$$

Thus, *TLF* yields the highest social welfare when $\alpha \in (0, 1)$. This completes the social welfare analysis since we discussed the case $\alpha \geq 1$ at the very beginning of the proof. □

D Proofs of Results for the Setup with Adoption Costs

We first present the optimal strategies under each of the business models separately. As the proofs require defining a lot of parameters, in the interest of avoiding notation abuse, we add a subscript D to some of the newly defined parameters (to distinguish from parameters used in the previous proofs)

Proposition D.1. *Under CE-PL model, in the presence of adoption costs, the firm's optimal pricing strategy, the corresponding profit, and ensuing social welfare are:*

	$0 < \alpha < 13 - 4\sqrt{10}$			$\alpha \geq 13 - 4\sqrt{10}$	
	(a) $0 \leq c < c^{\dagger}$	(b) $c^{\dagger} \leq c < 2\alpha$	$c \geq 2\alpha$	$0 \leq c < 2\alpha$	$c \geq 2\alpha$
p^*_{CE-PL}	$\frac{\alpha^2 c - c + 2\alpha\left(1+\alpha-\sqrt{(\alpha+1)(\alpha(c+2)-c)}\right)}{1-\alpha^2}$	$\frac{1}{2}(2\alpha - c)$	-	$\frac{1}{2}(2\alpha - c)$	-
π^*_{CE-PL}	$\frac{2\alpha+\alpha^2(c+6)-c-4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}{(1-\alpha)^2}$	$\frac{(c-2\alpha)^2}{8\alpha}$	-	$\frac{(c-2\alpha)^2}{8\alpha}$	-
SW^*_{CE-PL}	$\tilde{SW}_{CE-PL,E}$	$\frac{(c-2\alpha)((4\alpha-1)c-6\alpha)}{16\alpha^2}$	-	$\frac{(c-2\alpha)((4\alpha-1)c-6\alpha)}{16\alpha^2}$	-
Paid adoption	in both periods	only in period 1	none	only in period 1	none

where

$$\tilde{SW}_{CE-PL,D} = \frac{\left(8\alpha^3 + 8\alpha^2 + 4\alpha - \left(4\alpha^3 - 6\alpha + 2\right)c\right)\sqrt{(\alpha+1)(2\alpha - (1-\alpha)c)} + 2\alpha^4\left(c^2 + c - 4\right) - \alpha^2(5c(c+1) + 14) + 2\alpha(c+1)^2 + (c-1)c}{2(1-\alpha)^2(\alpha+1)(2\alpha - (1-\alpha)c)},$$

and threshold $c^{\dagger}(\alpha)$ is the unique solution to the equation $\Phi_{PL,D}(\alpha, c) = 0$ over the interval $\left(c_L^{\dagger}, \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}\right)$, with

$$\begin{aligned}\Phi_{PL,D}(\alpha, c) \triangleq\ & (1-\alpha)^4 c^4 \\ & + 8(1-\alpha)^2(\alpha(2-3\alpha)+1)\alpha c^3 \\ & + \left(16(\alpha(2-3\alpha)+1)^2\alpha^2 + 8(\alpha-1)^2((\alpha-14)\alpha-3)\alpha^2\right)c^2 \\ & + \left(32\alpha^3(\alpha(2-3\alpha)+1)((\alpha-14)\alpha-3) - 1024(\alpha-1)\alpha^4(\alpha+1)\right)c \\ & - 2048(\alpha+1)\alpha^5 + 16\alpha^4((\alpha-14)\alpha-3)^2,\end{aligned}$$

and

$$c_L^{\dagger} \triangleq \begin{cases} \left(6 - \frac{4}{1-\alpha}\right)\alpha, & \text{if } 0 < \alpha < \frac{1}{3}, \\ 0, & \text{if } \frac{1}{3} \leq \alpha \leq 13 - 4\sqrt{10}. \end{cases}$$

Proof. In period 1, consumers with type θ purchase the product iff $2\alpha\theta - c \geq p$. To make any profit, the firm is constrained to trigger adoption in period 1 (otherwise, no customer would update their priors and there will also be no adopters in period 2 either). To achieve that, the firm has to set price $p \in (0, 2\alpha - c)$. Thus, it immediately follows that the firm can make profit iff $0 \leq c < 2\alpha$. As such, the firm does not enter the market if $c \geq 2\alpha$.

In the remaining part of the proof we focus on the scenario $0 \leq c < 2\alpha$. In period 1, the marginal adopter has type $\theta_1 = \frac{c+p}{2\alpha}$ and the installed base is $N_1 = 1 - \theta_1 = 1 - \frac{c+p}{2\alpha}$.

At the beginning of period 2, the consumers who did not adopt in period 1 update their priors via social learning from $a_1 = \alpha$ to:

$$a_2 = a_1 + (1 - a_1)\, N_1 = \frac{1}{2}\left(2 + c + p - \frac{c+p}{\alpha}\right).$$

In period 2, new consumers purchase the product if their type θ satisfies $a_2\theta - c \geq p$. The marginal potential consumer in period 2 has type $\theta_2 = \frac{c+p}{\frac{1}{2}\left(2+c+p-\frac{c+p}{\alpha}\right)}$. We have new adopters in period 2 iff $0 \leq \theta_2 < \theta_1$. We have two cases:

- Case 1: $0 < \alpha < 1$.
  In this case, we have three subcases:
  - Case 1-i: $0 \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$, $0 < p < \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$.
    Then we have $0 < \theta_2 < \theta_1$. Then, $N_2 = \theta_1 - \theta_2 > 0$. In this case, the firm's profit maximization problem becomes:

    $$\max_{0<p<\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}} \pi_{CE-PL} = \max_{0<p<\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}} p\left(1 - \frac{c+p}{\frac{1}{2}\left(2+c+p-\frac{c+p}{\alpha}\right)}\right).$$

    It can be shown that $\frac{\partial^2 \pi_{CE-PL}}{\partial p^2} < 0$ for $p \in \left(0, \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}\right)$. Thus, it is sufficient to solve FOC:

    $$\frac{\partial \pi_{CE-PL}}{\partial p} = \frac{-\alpha^2\left((c+p)^2 + 4(p-1)\right) - 4\alpha(c+p) + (c+p)^2}{(c+p-\alpha(c+p+2))^2} = 0.$$

    Without constrains, the FOC yields two solutions:

    $$p_{1,D} = \frac{\alpha^2 c - c + 2\alpha\left(1+\alpha+\sqrt{(\alpha+1)(\alpha(c+2)-c)}\right)}{1-\alpha^2},$$
    $$p_{2,D} = \frac{\alpha^2 c - c + 2\alpha\left(1+\alpha-\sqrt{(\alpha+1)(\alpha(c+2)-c)}\right)}{1-\alpha^2}.$$

    It can be shown that $p_{1,D} > \max\left\{\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}, p_{2,D}\right\}$ and $p_{2,D} > 0$. Comparing $p_{2,D}$ with $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, we have three subcases:
    * Case 1-i-a: $0 < \alpha < \frac{1}{2}\left(\sqrt{3}-1\right)$, $0 < c < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$.
      Then $0 < p_{2,D} < \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, and it immediately follows that $p^*_{CE-PL} = p_{2,D} = \frac{\alpha^2c-c+2\alpha\left(1+\alpha-\sqrt{(\alpha+1)(\alpha(c+2)-c)}\right)}{1-\alpha^2}$, and $\pi^*_{CE-PL} = \frac{2\alpha+\alpha^2(c+6)-c-4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}{(1-\alpha)^2}$.
    * Case 1-i-b: $0 < \alpha < \frac{1}{2}\left(\sqrt{3}-1\right)$, $\frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha} \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$.
      Then $p_{2,D} \geq \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$. In this case, we have $p^*_{CE-PL} \to \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$. This case is suboptimal as optimal pricing is pushed into case 1-ii.
    * Case 1-i-c: $\frac{1}{2}\left(\sqrt{3}-1\right) \leq \alpha < 1$.
      Then $p_{2,D} \geq \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$. In this case, we have $p^*_{CE-PL} \to \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, and $\pi^*_{CE-PL} = \frac{\alpha(2\alpha(1-2\alpha)-(1-\alpha)c)}{(1-\alpha)^2}$. This case is suboptimal as optimal pricing is pushed into case 1-ii.

– Case 1-ii: $0 \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$, $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha} \leq p < 2\alpha - c$.
Then we have $\theta_2 \geq \theta_1$. In this case, $N_2 = 0$; adoption takes place only in period 1. The firm's profit maximization problem becomes:

$$\max_{\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha} \leq p < 2\alpha-c} \pi_{CE-PL} = \max_{\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha} \leq p < 2\alpha-c} p\left(1 - \frac{c+p}{2\alpha}\right).$$

Since the profit function is quadratic concave in p, it is sufficient to use FOC. Unconstrained, FOC yields the following solution:

$$p_{3,D} = \frac{1}{2}(2\alpha - c).$$

It is obvious that $p_{3,D} < 2\alpha - c$. Comparing $p_{3,D}$ with $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, we have three subcases:

  * Case 1-ii-a: $0 < \alpha < \frac{1}{3}$, $0 \leq c < \left(6 - \frac{4}{1-\alpha}\right)\alpha$.
  Then $0 < p_{3,D} < \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$. Then, we have the corner solution $p^*_{CE-PL} = \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, which is dominated by case 1-i-a (at the corner solution we have $\theta_1 = \theta_2$).
  * Case 1-ii-b: $0 < \alpha < \frac{1}{3}$, $\left(6 - \frac{4}{1-\alpha}\right)\alpha \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$.
  Then $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha} \leq p_3 < 2\alpha - c$. Thus, $p^*_{CE-PL} = p_{3,D} = \frac{1}{2}(2\alpha - c)$ and $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$.
  * Case 1-ii-c: $\frac{1}{3} \leq \alpha < 1$, $0 \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$.
  Then $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha} \leq p_{3,D} < 2\alpha - c$. Thus, $p^*_{CE-PL} = p_{3,D} = \frac{1}{2}(2\alpha - c)$ and $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$.

– Case 1-iii: $\frac{2\alpha-4\alpha^2}{1-\alpha} \leq c < 2\alpha$.
Then $\theta_2 \geq \theta_1$. In this case, $N_2 = 0$; adoption takes place only in period 1. The firm's profit maximization problem becomes:

$$\max_{0<p<2\alpha-c} \pi_{CE-PL} = \max_{0<p<2\alpha-c} p\left(1 - \frac{c+p}{2\alpha}\right).$$

Since the profit function is quadratic concave in p, it is sufficient to use FOC. Unconstrained, FOC yields the same solution $p_{3,D} = \frac{1}{2}(2\alpha - c)$. Then, $p^*_{CE-PL} = p_3 = \frac{1}{2}(2\alpha - c)$ and $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$.

As cases 1-i-b, 1-i-c, and 1-ii-a are suboptimal, in order to determine the optimal strategy when $0 \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$ we are left to compare cases 1-i-a to cases and 1-ii-b and 1-ii-c. When $0 < \alpha < \frac{1}{2}\left(\sqrt{3} - 1\right)$ we have $\left(6 - \frac{4}{1-\alpha}\right)\alpha < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha} < \frac{2\alpha-4\alpha^2}{1-\alpha}$. Thus, we only need to explore two subregions:

– Comparison Subregion 1: $0 < \alpha < \frac{1}{3}$, $\left(6 - \frac{4}{1-\alpha}\right)\alpha \leq c < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$.

In this region, denote the difference between profits under case 1-i-a and case 1-ii-b as:

$$\Delta_{PL,D}(\alpha,c) \triangleq \frac{2\alpha+\alpha^2(c+6)-c-4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}{(1-\alpha)^2} - \frac{(c-2\alpha)^2}{8\alpha}.$$

Note that:

$$\Delta_{PL,D}(\alpha,c)>0 \iff 4\alpha^2((14-\alpha)\alpha+3)-(1-\alpha)^2-c^2-4\alpha(3\alpha+1)(1-\alpha)c > 32\alpha^2\sqrt{(\alpha+1)(2\alpha-(1-\alpha)c)}.$$

It can be shown that the $4\alpha^2((14-\alpha)\alpha+3)-(1-\alpha)^2-c^2-4\alpha(3\alpha+1)(1-\alpha)c>0$. Thus, the sign of $\Delta_{PL,D}(\alpha,c)$ is same as the sign of $\Phi_{PL,D}(\alpha,c)$, where:

$$\begin{aligned}\Phi_{PL,D}(\alpha,c) &\triangleq \left(4\alpha^2((14-\alpha)\alpha+3)-(1-\alpha)^2-c^2-4\alpha(3\alpha+1)(1-\alpha)c\right)^2 \\ &\quad -1024\alpha^4(\alpha+1)(2\alpha-(1-\alpha)c). \\ &=(1-\alpha)^4c^4 \\ &\quad +8(1-\alpha)^2(\alpha(2-3\alpha)+1)\alpha c^3 \\ &\quad +\left(16(\alpha(2-3\alpha)+1)^2\alpha^2+8(\alpha-1)^2((\alpha-14)\alpha-3)\alpha^2\right)c^2 \\ &\quad +\left(32\alpha^3(\alpha(2-3\alpha)+1)((\alpha-14)\alpha-3)-1024(\alpha-1)\alpha^4(\alpha+1)\right)c \\ &\quad -2048(\alpha+1)\alpha^5+16\alpha^4((\alpha-14)\alpha-3)^2.\end{aligned} \tag{D.1}$$

It can be shown that, in this region, $\frac{\partial\Phi_{PL,D}(\alpha,c)}{\partial c}<0$. Next, we check the sign of $\Phi_{PL,D}(\alpha,c)$ at the two extremes in c :

$$\begin{aligned}\Phi_{PL,D}(\alpha,c)\Big|_{c=\left(6-\frac{4}{1-\alpha}\right)} &= 256(1-2\alpha)^2\alpha^6>0, \\ \Phi_{PL,D}(\alpha,c)\Big|_{c=\frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}} &= 16(1-2\alpha)^2\alpha^7(\alpha(4(\alpha-1)\alpha-31)-32)<0.\end{aligned}$$

Thus, there exists a unique solution $c=c^\dagger(\alpha)$ to the equation $\Delta_{PL,D}(\alpha,c)=0$ over the interval $\left(\left(6-\frac{4}{1-\alpha}\right),\frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}\right)$, such that when $\left(6-\frac{4}{1-\alpha}\right)\le c<c^\dagger$, case 1-i-a dominates case 1-ii-b; when $c^\dagger\le c<\frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$, case 1-ii-b dominates case 1-i-a.

– Comparison Subregion 2: $\frac{1}{3}\le\alpha<\frac{1}{2}\left(\sqrt{3}-1\right)$, $0\le c<\frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$.
In this region, the difference between profits under case 1-i-a and case 1-ii-c is again given by $\Phi_{PL,D}(\alpha,c)$, as defined in equation (D.1). Following the same steps as above, we can show that in this region as well we have $\frac{\partial\Phi_{PL,D}(\alpha,c)}{\partial c}<0$ and:

$$\Phi_{PL,D}(\alpha,c)\Big|_{c=\frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}} = 16(1-2\alpha)^2\alpha^7(\alpha(4(\alpha-1)\alpha-31)-32)<0.$$

Then, we look at the other extreme in c:

$$\Phi_{PL,D}(\alpha,c)\Big|_{c=0}=16(1-\alpha)^2\alpha^4((\alpha-26)\alpha+9).$$

It can be shown that:

$$\Phi_{PL,D}(\alpha, c)\Big|_{c=0} \begin{cases} > 0 & , \frac{1}{3} \leq \alpha < 13 - 4\sqrt{10}, \\ \leq 0 & , 13 - 4\sqrt{10} < \alpha < \frac{1}{2}\left(\sqrt{3} - 1\right). \end{cases} \tag{D.2}$$

Thus:

* When $\frac{1}{3} \leq \alpha < 13 - 4\sqrt{10}$, there exists a unique solution $c = c^{\dagger}(\alpha)$[D-1] to the equation $\Delta_{PL,D}(\alpha, c) = 0$ over the interval $\left(0, \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}\right)$, such that when $0 \leq c < c^{\dagger}$, case 1-i-a dominates case 1-ii-c; when $c^{\dagger} \leq c < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$, case 1-ii-c dominates case 1-i-a;
* When $13 - 4\sqrt{10} < \alpha < \frac{1}{2}\left(\sqrt{3} - 1\right)$, case 1-ii-c dominates case 1-i-a.

In summary, in case 1, when $0 < \alpha < 13 - 4\sqrt{10}$ and $0 \leq c < c^{\dagger}(\alpha)$, then we have $p^*_{CE-PL} = \frac{\alpha^2 c - c + 2\alpha\left(1 + \alpha - \sqrt{(\alpha+1)(\alpha(c+2)-c)}\right)}{1-\alpha^2}$, $\pi^*_{CE-PL} = \frac{2\alpha + \alpha^2(c+6) - c - 4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}{(1-\alpha)^2}$, and

$$SW^*_{CE-PL} = \frac{\left(8\alpha^3 + 8\alpha^2 + 4\alpha - \left(4\alpha^3 - 6\alpha + 2\right)c\right)\sqrt{(\alpha+1)(2\alpha - (1-\alpha)c)} + 2\alpha^4\left(c^2 + c - 4\right) - \alpha^2(5c(c+1) + 14) + 2\alpha(c+1)^2 + (c-1)c}{2(1-\alpha)^2(\alpha+1)(2\alpha - (1-\alpha)c)}.$$

Otherwise, $p^*_{CE-PL} = \frac{1}{2}(2\alpha - c)$, $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$, and $SW^*_{CE-PL} = \frac{(c-2\alpha)((4\alpha-1)c-6\alpha)}{16\alpha^2}$.

- Case 2: $\alpha \geq 1$.
  In this case, $a_1 > a_2 > a = 1$. None of the period 1 non-adopters will purchase in period 2. The profit maximization problem becomes:

$$\max_{0<p<2\alpha-c} \pi_{CE-PL} = \max_{0<p<2\alpha-c} p\left(1 - \frac{c+p}{2\alpha}\right).$$

Then, $p^*_{CE-PL} = \frac{1}{2}(2\alpha - c)$, $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$, and $SW^*_{CE-PL} = \frac{(c-2\alpha)((4\alpha-1)c-6\alpha)}{16\alpha^2}$. □

Proposition D.2. *Under CE-SUB model, in the presence of adoption costs, the firm's optimal pricing strategy, the corresponding profit, and ensuing social welfare are:*

- $0 < \alpha \leq 1$.

	$0 \leq c < \alpha$	$\alpha \leq c$
p^*_{CE-SUB}	$p_{a,D}$	-
π^*_{CE-SUB}	$\pi_{1,CE-SUB,D}$	-
SW^*_{CE-SUB}	$SW_{1,CE-SUB,D}$	-
Paid adoption	in both periods	none

where $p_{a,D}$ is the unique solution to the equation $G_{SUB,D}(p) = 0$ over the interval $\left[\frac{\alpha-c}{2}, \alpha - c\right)$

[D-1] We use the same notation as in the prior case, since the solution is to the same equation, but over a different range of α.

with

$$\begin{aligned} G_{SUB,D}(p) \triangleq & -2(1-\alpha)^2 p^3 + p^2(1-\alpha)((6-\alpha)\alpha + 5(\alpha-1)c) \\ & + 2p\left((\alpha-3)\alpha^2 - 2(\alpha-1)^2 c^2 + \alpha\left(\alpha^2 - 6\alpha + 5\right)c\right) \\ & + (1-\alpha)^2 c^3 + 2\alpha^3 + (3\alpha-5)\alpha^2 c + \alpha\left(\alpha^2 - 5\alpha + 4\right)c^2, \end{aligned}$$

and

$$\pi_{1,CE-SUB,D} = p_{a,D}\left(2 - \frac{c+p_{a,D}}{\alpha} - \frac{c+p_{a,D}}{1+c+p_{a,D} - \frac{c+p_{a,D}}{\alpha}}\right),$$

$$SW_{1,CE-SUB,D} = 1 - c - \frac{(c+p_{a,D})^2}{2\alpha^2} - \frac{(c+p_{a,D})^2}{2\left(1+c+p_{a,D} - \frac{c+p_{a,D}}{\alpha}\right)^2} + \frac{c(c+p_{a,D})}{1+c+p_{a,D} - \frac{c+p_{a,D}}{\alpha}}.$$

- $1 < \alpha \leq 2$.

	$0 \leq c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$	$\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$	$\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$	$\alpha \leq c$
p^*_{CE-SUB}	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{c}{\alpha-1}$	$\frac{\alpha-c}{2}$	-
π^*_{CE-SUB}	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{2c(\alpha-c-1)}{(\alpha-1)^2}$	$\frac{(\alpha-c)^2}{2\alpha}$	-
SW^*_{CE-SUB}	$SW_{2,CE-SUB,D}$	$\frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$	$\frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$	-
Paid adoption	in period 1	in both periods	in both periods	none

where $SW_{2,CE-SUB,D} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.

- $2 < \alpha < \frac{1}{2}\left(\sqrt{17}+3\right)$.

	$0 \leq c < \alpha - 2$			$\alpha - 2 \leq c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$	$\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$	$\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$	$\alpha \leq c$
	$2 < \alpha \leq 3$	$3 < \alpha < \frac{1}{2}\left(\sqrt{17}+3\right)$					
		$0 \leq c < c_{1,SUB,D}$	$c_{1,SUB,D} \leq c < \alpha - 2$				
p^*_{CE-SUB}	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{\alpha-c}{2}$	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{c}{\alpha-1}$	$\frac{\alpha-c}{2}$	-
π^*_{CE-SUB}	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{(\alpha-c)^2}{4\alpha}$	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{2c(\alpha-c-1)}{(\alpha-1)^2}$	$\frac{(\alpha-c)^2}{2\alpha}$	-
SW^*_{CE-SUB}	$SW_{2,CE-SUB,D}$	$\frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$	$SW_{2,CE-SUB,D}$	$SW_{2,CE-SUB,D}$	$\frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$	$\frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$	-
Paid adoption	in both periods	in period 1	in both periods	in both periods	in both periods	in both periods	none

where $c_{1,SUB,D} = \alpha - \sqrt{\alpha+1} - 1$.

- $\frac{1}{2}\left(\sqrt{17}+3\right) \leq \alpha < 4\sqrt{2} + 5$.

	$0 \leq c < c_{1,SUB,D}$	$c_{1,SUB,D} \leq c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$	$\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$	$\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$	$\alpha \leq c$
p^*_{CE-SUB}	$\frac{\alpha-c}{2}$	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{c}{\alpha-1}$	$\frac{\alpha-c}{2}$	-
π^*_{CE-SUB}	$\frac{(\alpha-c)^2}{4\alpha}$	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{2c(\alpha-c-1)}{(\alpha-1)^2}$	$\frac{(\alpha-c)^2}{2\alpha}$	-
SW^*_{CE-SUB}	$\frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$	$SW_{2,CE-SUB,D}$	$\frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$	$\frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$	-
Paid adoption	in period 1	in both periods	in both periods	in both periods	none

- $4\sqrt{2}+5 \leq \alpha$.

	$0 \leq c < c_{3,SUB,D}$	$c_{3,SUB,D} \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$	$\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$	$\alpha \leq c$
p^*_{CE-SUB}	$\frac{\alpha-c}{2}$	$\frac{c}{\alpha-1}$	$\frac{\alpha-c}{2}$	-
π^*_{CE-SUB}	$\frac{(\alpha-c)^2}{4\alpha}$	$\frac{2c(\alpha-c-1)}{(\alpha-1)^2}$	$\frac{(\alpha-c)^2}{2\alpha}$	-
SW^*_{CE-SUB}	$\frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$	$\frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$	$\frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$	-
Adoption	in period 1	in both periods	in both periods	none

where $c_{3,SUB,D} = \frac{(\alpha-1)\alpha(\alpha+3)-2\sqrt{2}\alpha(\alpha-1)}{\alpha(\alpha+6)+1}$.

Proof. In period 1, customers subscribe iff $\alpha\theta - c \geq p$. To make profit, the firm is constrained to set $0 < p < \alpha - c$. Thus, it immediately follows that the firm can make profit iff $0 \leq c < \alpha$. As such, the firm does not enter the market if $c \geq \alpha$.

In the remaining part of the proof we focus on the scenario $0 \leq c < \alpha$. In period 1, the marginal adopter has type $\theta_1 = \frac{c+p}{\alpha}$ and the installed base is $N_1 = 1 - \theta_1 = 1 - \frac{c+p}{\alpha}$.

At the beginning of period 2, the consumers who did not adopt in period 1 update their priors via social learning from $a_1 = \alpha$ to:

$$a_2 = a_1 + (1-a_1)N_1 = 1 + p + c - \frac{c+p}{\alpha}.$$

In period 2, *new* consumers subscribe to the product/service if their type θ satisfies $a_2\theta - c \geq p$.

We have two cases:

- Case 1: $0 < \alpha \leq 1$.
  In this case, $a_1 \leq a_2 \leq a = 1$. The marginal customer type for period 1 non-adopters at the beginning of period 2 is $\theta_2 = \frac{c+p}{1+c+p-\frac{c+p}{\alpha}} < \theta_1$. Thus, all customers with types $\theta \in [\theta_2, \theta_1)$ are new adopters in period 2 (i.e., fresh subscribers). In the case of period 1 adopters (i.e., with type $\theta \in [\theta_1, 1]$), their valuation of the product updates upwards and there is no more adoption cost in period 2 (since adoption cost is a one-time cost). Thus, all adopters in period 1 continue to subscribe in period 2. The profit maximization problem becomes:

$$\max_{0<p<\alpha-c} \pi_{CE-SUB} = \max_{0<p<\alpha-c} p(1-\theta_1+1-\theta_2) = \max_{0<p<\alpha-c} p\left(2 - \frac{c+p}{\alpha} - \frac{c+p}{1+c+p-\frac{c+p}{\alpha}}\right).$$

  It can be shown that $\frac{\partial^2 \pi_{CE-SUB}}{\partial p^2} < 0$. Hence, FOC is sufficient to determine the optimal price. We have:

$$\frac{\partial \pi_{CE-SUB}}{\partial p} = \frac{G_{SUB,D}}{Q_{SUB,D}},$$

where:

$$
\begin{aligned}
G_{SUB,D}(p) \triangleq & -2(1-\alpha)^2p^3 + p^2(1-\alpha)((6-\alpha)\alpha + 5(\alpha-1)c) \\
& + 2p\left((\alpha-3)\alpha^2 - 2(\alpha-1)^2c^2 + \alpha\left(\alpha^2 - 6\alpha + 5\right)c\right) \\
& + (1-\alpha)^2c^3 + 2\alpha^3 + (3\alpha-5)\alpha^2c + \alpha\left(\alpha^2 - 5\alpha + 4\right)c^2, \\
Q_{SUB,D}(p) \triangleq & \ \alpha(\alpha - (1-\alpha)c - (1-\alpha)p)^2 > 0.
\end{aligned}
$$

Thus, when solving FOC $\left(\frac{\partial \pi_{CE-SUB}}{\partial p} = 0\right)$, it is enough to look at the numerator. We further have two cases:

- Case 1-i: $0 < \alpha < 1$.
  In this case, $G_{SUB,D}(p)$ is cubic in p and, thus, the equation $\frac{\partial G_{SUB,D}(p)}{\partial p} = 0$ has two solutions:

$$
p_{1,SUB,D} = \frac{\alpha + \alpha c - c}{1-\alpha} \quad \text{and} \quad p_{2,SUB,D} = \frac{-\alpha^2 + 3\alpha + 2\alpha c - 2c}{3(1-\alpha)}.
$$

  It can be shown that $p_{1,SUB,D} > \alpha - c$ and $p_{2,SUB,D} > \alpha - c$. Thus, $\frac{\partial G_{SUB,D}(p)}{\partial p} < 0$ for all $p \in (0, \alpha - c)$. Evaluating $G_{SUB,D}(p)$ at various threshold points, it can be shown that:

$$
G_{SUB,D}(0) > G_{SUB,D}\left(\frac{\alpha - c}{2}\right) > 0 > G_{SUB,D}(\alpha - c).
$$

  Thus, $G_{SUB,D}(p) = 0$ has a unique solution $p_{a,D} \in \left[\frac{\alpha-c}{2}, \alpha - c\right)$ over the real line, which is also the optimal profit-maximizing price in this region ($p^*_{CE-SUB} = p_{a,D}$). More precisely, $\frac{\partial \pi_{CE-SUB}(p)}{\partial p} > 0$ for $p \in (0, p_{a,D})$ and $\frac{\partial \pi_{CE-SUB}(p)}{\partial p} < 0$ for $p \in (p_{a,D}, \alpha - c)$. The formulas for the optimal profit and associated social welfare follow trivially.
- Case 1-ii: $\alpha = 1$.
  In this case, $G_{SUB,D}(p) = -2(c + 2p - 1)$. The equation $G_{SUB,D}(p) = 0$ has a unique solution $p_{a,D} = \frac{1-c}{2} \in \left[\frac{1-c}{2}, 1 - c\right)$. Therefore, $p^*_{CE-SUB} = p_{a,D}$.

- Case 2: $1 < \alpha$.
  In this case, $a_1 \geq a_2 \geq a = 1$. None of period 1 non-adopters will subscribe in period 2 as they revise downwards their perceived valuation of the product. On the other hand, period 1 subscribers, when exploring renewing their subscription for period 2, have to consider the tension between two opposing forces: (i) the downgrading in the perceived valuation (which by now has been calibrated to the real value through experience learning) and (ii) the reduction in adoption cost (the adoption cost is incurred only at adoption time and, as such, returning customers would no longer incur that cost in period 2). Thus, the marginal adopting customer type in period 2, θ_2, satisfies $\theta_2 = \max\{\theta_1, \min\{1, p\}\}$. Since $0 < p < \alpha - c$, comparing $\alpha - c$ with 1, we get three cases:

  - Case 2-i: $0 \leq c < \alpha - 1$, $0 < p < 1$.
    In this case, $\alpha - c > 1 > p$ and $\theta_2 = \max\{\theta_1, p\}$. Comparing θ_1 and p, we obtain two sub-cases:

* Case 2-i-A: $0 < p \leq \frac{c}{\alpha-1}$.
In this case, $\theta_1 \geq p$ and all period 1 subscribers continue to subscribe in period 2. Thus, the profit maximization problem becomes:

$$\max_{0<p<\frac{c}{\alpha-1}} \pi_{CE-SUB} = \max_{0<p<\frac{c}{\alpha-1}} 2p\left(1 - \frac{c+p}{\alpha}\right).$$

Since the profit is quadratic concave in p, it is sufficient to use FOC to derive optimal price. Unconstrained, FOC yields the following solution:

$$p_{3,SUB,D} = \frac{\alpha - c}{2}.$$

Comparing $p_{3,SUB,D}$ with $\frac{c}{\alpha-1}$, we obtain two sub-cases:

  · Case 2-i-A-I: $0 \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$.
  In this case, $p_{3,SUB,D} > \frac{c}{\alpha-1}$. Then, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$ and $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$.
  · Case 2-i-A-II: $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha - 1$.
  In this case, $p_{3,SUB,D} \leq \frac{c}{\alpha-1}$. Then, $p^*_{CE-SUB} = p_{3,SUB,D} = \frac{\alpha-c}{2}$ and $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$.

* Case 2-i-B: $\frac{c}{\alpha-1} < p < 1$.
In this case, $\theta_2 = p > \theta_1$ and the profit maximization problem becomes:

$$\max_{\frac{c}{\alpha-1} \leq p<1} \pi_{CE-SUB} = \max_{\frac{c}{\alpha-1} \leq p<1} p\left(2 - \frac{c+p}{\alpha} - p\right).$$

Since the profit function is quadratic concave in p, it is sufficient to use FOC to identify the optimal price. Unconstrained, FOC yields the following solution:

$$p_{4,SUB,D} = \frac{2\alpha - c}{2(\alpha+1)} < 1.$$

Comparing $p_{4,SUB,D}$ with $\frac{c}{\alpha-1}$, we obtain two sub-cases:

  · Case 2-i-B-I: $0 \leq c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$.
  In this case, $p_{4,SUB,D} > \frac{c}{\alpha-1}$, $p^*_{CE-SUB} = p_{4,SUB,D} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$.
  · Case 2-i-B-II: $\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < \alpha - 1$.
  In this case, $p_{4,SUB,D} \leq \frac{c}{\alpha-1}$, and $p^*_{CE-SUB} \to \frac{c}{\alpha-1}$. This case is suboptimal as we are pushed into case 2-i-A.

Since $\frac{\alpha^2-\alpha}{\alpha+1} > \frac{2(\alpha-1)\alpha}{3\alpha+1}$, comparing case 2-i-A (both subcases) against case 2-i-B-I and reorganizing, we get:

* Case 2-i-a: $0 \leq c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$.
In this case, $\frac{2c(\alpha-c-1)}{(\alpha-1)^2} < \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$, i.e., case 2-i-B-I dominates case 2-i-A-I, $p^*_{CE-SUB} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$.

* Case 2-i-b: $\frac{2(\alpha-1)\alpha}{3\alpha+1} \le c < \frac{\alpha^2-\alpha}{\alpha+1}$.
  In this case, as discussed above, case 2-i-B-II is dominated by 2-i-A-I. Thus, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$.
* Case 2-i-c: $\frac{\alpha^2-\alpha}{\alpha+1} \le c < \alpha - 1$.
  In this case, as discussed above, case 2-i-B-II is dominated by 2-i-A-II, Thus $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$.

- Case 2-ii: $0 \le c < \alpha - 1$, $1 \le p < \alpha - c$
  In this case, $\theta_2 = 1$. There are no subscribers in period 2. The profit maximization problem becomes:

$$\max_{1 \le p < \alpha - c} \pi_{CE-SUB} = \max_{1 \le p < \alpha - c} p \left(1 - \frac{c+p}{\alpha}\right).$$

  Since the function is quadratic, it is sufficient to use FOC. Unconstrained, FOC yields the following solution:

$$p_{3,SUB,D} = \frac{\alpha - c}{2} < \alpha - c.$$

  Comparing $p_{3,SUB,D}$ with 1, we obtain three sub-cases:

  * Case 2-ii-a: $1 < \alpha \le 2$.
    In this case, $p_{3,SUB,D} \le 1$, and thus $p^*_{CE-SUB} = 1$, $\pi^*_{CE-SUB} = 1 - \frac{c+1}{\alpha}$.
  * Case 2-ii-b: $2 < \alpha$, $0 \le c < \alpha - 2$.
    In this case, $p_{3,SUB,D} > 1$, $p^*_{CE-SUB} = p_{3,SUB,D} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$.
  * Case 2-ii-c: $2 < \alpha$, $\alpha - 2 \le c < \alpha - 1$.
    In this case, $p_{3,SUB,D} \le 1$, $p^*_{CE-SUB} = 1$, $\pi^*_{CE-SUB} = 1 - \frac{c+1}{\alpha}$.

- Case 2-iii: $\alpha - 1 \le c < \alpha$, $0 < p < \alpha - c \le 1$.
  In this case, $\frac{c}{\alpha-1} \ge 1 > p$. Thus, $\theta_1 > p$ and $\theta_2 = \theta_1$. The profit maximization problem becomes:

$$\max_{0 \le p < \alpha - c} \pi_{CE-SUB} = \max_{0 \le p < \alpha - c} 2p \left(1 - \frac{c+p}{\alpha}\right).$$

  It follows that $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$.

Let us summarize case 2 (and in particular compare 2.i and 2.ii cases). It is easy to see that $\alpha - 2 < \frac{\alpha^2-\alpha}{\alpha+1}$. Comparing $\alpha - 2$ and $\frac{2(\alpha-1)\alpha}{3\alpha+1}$, we get three cases:

- $1 < \alpha \le 2$.
  It can be easily shown that case 2-i dominates case 2-ii-a when $c < \alpha - 1$. Combining with case 2-iii, we extend the region to $c < \alpha$.
- $2 < \alpha < \frac{1}{2}\left(\sqrt{17} + 3\right)$.
  In this case, we have $\alpha - 2 < \frac{2(\alpha-1)\alpha}{3\alpha+1} < \frac{\alpha^2-\alpha}{\alpha+1}$. We further have four sub-cases:

* $0 \le c < \alpha - 2$.
In this region, denote the profit difference between case 2-i-a and case 2-ii-b as:

$$H_{1,SUB,D} \triangleq \frac{(2\alpha - c)^2}{4\alpha(\alpha+1)} - \frac{(\alpha - c)^2}{4\alpha} = \frac{-c^2 + 2(\alpha-1)c - (\alpha-3)\alpha}{4(\alpha+1)}.$$

The equation $H_{1,SUB,D} = 0$ has two solutions:

$$c_{1,SUB,D} = \alpha - \sqrt{\alpha+1} - 1 \quad \text{and} \quad c_{2,SUB,D} = \alpha + \sqrt{\alpha+1} - 1.$$

We have $c_{1,SUB,D} < \alpha - 2 < c_{2,SUB,D}$. Comparing $c_{1,SUB,D}$ with 0, we get:
  · If $2 < \alpha \le 3$, then $c_{1,SUB,D} < 0$ and $H_{1,SUB,D} \ge 0$ for all $c \in [0, \alpha-2)$, i.e. case 2-i-a dominates case 2-ii-b. Thus, $p^*_{CE-SUB} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$, and $SW^*_{CE-SUB} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.
  · If $3 < \alpha < \frac{1}{2}\left(\sqrt{17}+3\right)$ and $0 \le c < c_{1,SUB,D} = \alpha - \sqrt{\alpha+1} - 1$, then $H_{1,SUB,D} < 0$, i.e. case 2-ii-b dominates case 2-i-a. Thus, $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$, $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$.
  · If $3 < \alpha < \frac{1}{2}\left(\sqrt{17}+3\right)$ and $c_{1,SUB,D} \le c < \alpha - 2$, then $H_{1,SUB,D} \ge 0$, i.e. case 2-i-a dominates case 2-ii-b. Thus, $p^*_{CE-SUB} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$, and $SW^*_{CE-SUB} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.

* $\alpha - 2 \le c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$.
In this region, it can be shown that $\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-a dominates case 2-ii-c. Thus, $p^*_{CE-SUB} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(c-2\alpha)^2}{4\alpha(\alpha+1)}$, and $SW^*_{CE-SUB} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.

* $\frac{2(\alpha-1)\alpha}{3\alpha+1} \le c < \frac{\alpha^2-\alpha}{\alpha+1}$.
In this region, it can be shown that $\frac{2c(\alpha-c-1)}{(\alpha-1)^2} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-b dominates case 2-ii-c. Thus, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$, and $SW^*_{CE-SUB} = \frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$.

* $\frac{\alpha^2-\alpha}{\alpha+1} \le c < \alpha$.
In this region, it can be shown that, when $\frac{\alpha^2-\alpha}{\alpha+1} \le c < \alpha - 1$, we have $\frac{(\alpha-c)^2}{2\alpha} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-c dominates case 2-ii-c, and $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$, $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$. Combining with case 2-iii, we extend the region to $\frac{\alpha^2-\alpha}{\alpha+1} \le c < \alpha$.

– $\alpha \ge \frac{1}{2}\left(\sqrt{17}+3\right)$.
In this case, we have $\frac{2(\alpha-1)\alpha}{3\alpha+1} \le \alpha - 2 < \frac{\alpha^2-\alpha}{\alpha+1}$. We further have four sub-cases:

* $0 \le c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$.
Following the same steps as in the above case, it can be shown that $0 < c_{1,SUB,D}$ and $\frac{2(\alpha-1)\alpha}{3\alpha+1} < c_{2,SUB,D}$. It can be shown that $c_{1,SUB,D} < \frac{2(\alpha-1)\alpha}{3\alpha+1}$ iff $\alpha < 4\sqrt{2} + 5$. We have the following sub-cases:

- If $\frac{1}{2}\left(\sqrt{17}+3\right) \leq \alpha < 4\sqrt{2}+5$, $0 \leq c < c_{1,SUB,D}$, case 2-ii-b dominates case 2-i-a. Thus, $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$, and $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$.
- If $\frac{1}{2}\left(\sqrt{17}+3\right) \leq \alpha < 4\sqrt{2}+5$ and $c_{1,SUB,D} \leq c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$, case 2-i-a dominates case 2-ii-b. Thus, $p^*_{CE-SUB} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$, $SW^*_{CE-SUB} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.
- If $\alpha \geq 4\sqrt{2}+5$, then $c_{1,SUB,D} \geq \frac{2(\alpha-1)\alpha}{3\alpha+1}$. Case 2-ii-b dominates case 2-i-a. $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$, and $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$.

* $\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < \alpha-2$.

  In this region, denote the profit difference between case 2-i-b and case 2-ii-b as:

$$\begin{aligned} H_{2,SUB,D} &\triangleq \frac{2c(\alpha-c-1)}{(\alpha-1)^2} - \frac{(\alpha-c)^2}{4\alpha} \\ &= \frac{-\left(\alpha^2+6\alpha+1\right)c^2+2\alpha\left(\alpha^2+2\alpha-3\right)c-(\alpha-1)^2\alpha^2}{4(\alpha-1)^2\alpha}. \end{aligned}$$

  The equation $H_{2,SUB,D}=0$ has two solutions:

$$\begin{aligned} c_{3,SUB,D} &= \frac{(\alpha-1)\alpha(\alpha+3)-2\sqrt{2}\alpha(\alpha-1)}{\alpha(\alpha+6)+1}, \\ c_{4,SUB,D} &= \frac{(\alpha-1)\alpha(\alpha+3)+2\sqrt{2}\alpha(\alpha-1)}{\alpha(\alpha+6)+1}. \end{aligned}$$

  It can be shown that $c_{3,SUB,D} < \alpha-2 < c_{4,SUB,D}$. We have $c_{3,SUB,D} < \frac{2(\alpha-1)\alpha}{3\alpha+1}$ iff $\alpha < 4\sqrt{2}+5$. We get the following sub-cases:
  - If $\frac{1}{2}\left(\sqrt{17}+3\right) \leq \alpha < 4\sqrt{2}+5$, then $c_{3,SUB,D} < \frac{2(\alpha-1)\alpha}{3\alpha+1}$. Thus, $H_{2,SUB,D} > 0$ for all $c \in \left[\frac{2(\alpha-1)\alpha}{3\alpha+1}, \alpha-2\right)$. Case 2-i-b dominates case 2-ii-b. We have $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$, and $SW^*_{CE-SUB} = \frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$.
  - If $\alpha \geq 4\sqrt{2}+5$ and $\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < c_{3,SUB,D}$, then case 2-ii-b dominates case 2-i-b. Thus, $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$, and $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$.
  - If $\alpha \geq 4\sqrt{2}+5$ and $c_{3,SUB,D} \leq c < \alpha-2$, then case 2-i-b dominates case 2-ii-b. Thus, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$, and $SW^*_{CE-SUB} = \frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$.

* $\alpha-2 \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$.

  In this region, it can be shown that $\frac{2c(\alpha-c-1)}{(\alpha-1)^2} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-b dominates case 2-ii-c. Thus, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$, and $SW^*_{CE-SUB} = \frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$.

* $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$.

  In this region, it can be shown that when $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha-1$, $\frac{(\alpha-c)^2}{2\alpha} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-c dominates case 2-ii-c and $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$,

$SW^*_{CE-SUB} = \frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$. Combining with case 2-iii, we extend the region to $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$. □

Proposition D.3. *Under* TLF *model, in the presence of adoption costs, the firm's optimal pricing strategy, the corresponding profit, and ensuing social welfare are:*

	$0 < \alpha < 1$			$\alpha \geq 1$		
	$0 \leq c < \frac{\alpha}{2}$	$\frac{\alpha}{2} \leq c < \alpha$	$c \geq \alpha$	$0 \leq c < \frac{\alpha}{2}$	$\frac{\alpha}{2} \leq c < \alpha$	$c \geq \alpha$
p^*_{TLF}	$\frac{1}{2}$	$\frac{c}{\alpha}$	-	$\frac{1}{2}$	$\frac{c}{\alpha}$	-
π^*_{TLF}	$\frac{1}{4}$	$\frac{c\left(1-\frac{c}{\alpha}\right)}{\alpha}$	-	$\frac{1}{4}$	$\frac{c\left(1-\frac{c}{\alpha}\right)}{\alpha}$	-
SW^*_{TLF}	$\frac{\alpha c^2(\alpha+2(\alpha-1)c)}{2(\alpha+(\alpha-1)c)^2} - c + \frac{7}{8}$	$\frac{1}{2}c\left(\frac{(2\alpha-1)c}{\alpha^2} - 2\right) + \frac{7}{8}$	-	$\frac{2\alpha^4-(1-\alpha)^2c^4+2(1-\alpha)^2\alpha c^3+\left(\alpha^2-\alpha^4\right)c^2+2(\alpha-2)\alpha^3 c}{2\alpha^2(\alpha+\alpha c-c)^2}$	$\frac{(\alpha-c)(-(\alpha-1)c+\alpha)}{\alpha^2}$	-
Paid adoption	in both periods	in both periods	none	in both periods	in both periods	none

Proof. Under TLF, all customers get the product for free in period 1, but they incur adoption cost c. Thus, customers of type θ start the free trial iff $\alpha\theta \geq c$. It is straightforward to see that there is no adoption if $0 < \alpha \leq c$. In the remaining part of the proof, we focus on the more interesting scenario in which adoption can take place, i.e. $0 \leq c < \alpha$.
The marginal adopter in period 1 has type $\theta_1 = \frac{c}{\alpha}$. The size of the adopter population in period 1 is $N_1 = 1 - \frac{c}{\alpha}$.
At the beginning of period 2, adopters in period 1 purchase the product iff $\theta \geq p$ (they already incurred the one-time adoption cost during the free trial in period 1). Period 1 adopters, through WOM, will help the non-adopters update their priors at the beginning of period 2 - however, the consumers who did not adopt in period 1 still have one free-trial period available to them and thus, regardless of how they update their priors, they will not contribute revenue to the firm. Hence, the only revenue can come from consumers who took advantage of the free trial in period 1. Thus, the firm must set $p \in (0, 1)$. The marginal *paying* customer in period 2 has type $\theta_2 = \max\{\theta_1, p\}$. Comparing p with θ_1, we get two cases:

- Case 1: $0 < p < \frac{c}{\alpha} = \theta_1$.
  In this case, $\theta_2 = \theta_1$ and the profit maximization problem becomes:
  $$\max_{0<p<\frac{c}{\alpha}} \pi_{TLF} = \max_{0<p<\frac{c}{\alpha}} p\left(1 - \frac{c}{\alpha}\right).$$
  It follows that $p^*_{TLF} \uparrow \frac{c}{\alpha}$. This case is suboptimal as p^*_{TLF} is pushed into case 2 region.
- Case 2: $\frac{c}{\alpha} \leq p < 1$.
  In this case, $\theta_2 \geq \theta_1$ and the profit maximization problem becomes:
  $$\max_{\frac{c}{\alpha}\leq p<1} \pi_{TLF} = \max_{\frac{c}{\alpha}\leq p<1} p(1-p).$$
  We have two subcases:
  - Case 2-i: If $0 \leq c < \frac{\alpha}{2}$, $p^*_{TLF} = \frac{1}{2}$, $\pi^*_{TLF} = \frac{1}{4}$.
  - Case 2-ii: If $\frac{\alpha}{2} \leq c < \alpha$, $p^*_{TLF} = \frac{c}{\alpha}$, $\pi^*_{TLF} = \frac{c}{\alpha}\left(1 - \frac{c}{\alpha}\right)$.

To get the social welfare, we further consider the non-adopters in period 1. Non-adopters in period 1 update their priors via social learning from $a_1 = \alpha$ to:

$$a_2 = a_1 + N_1(1 - a_1) = 1 + c - \frac{c}{\alpha}.$$

For a period 1 non-adopter of type $\theta < \theta_1$ to adopt in period 2 under free trial, it must be the case that $\theta \geq \tilde{\theta}_2 \triangleq \frac{c}{1+c-\frac{c}{\alpha}}$. Comparing $\tilde{\theta}_2$ with θ_1, we further split cases 2-i and 2-ii each into two subcases as follows:

- Case 2-i-a: $0 \leq c < \frac{\alpha}{2}$, $0 < \alpha < 1$.
  In this case, $\tilde{\theta}_2 < \theta_1$, $SW^*_{CE-SUB} = \frac{\alpha c^2(\alpha+2(\alpha-1)c)}{2(\alpha+(\alpha-1)c)^2} - c + \frac{7}{8}$.
- Case 2-i-b: $0 \leq c < \frac{\alpha}{2}$, $\alpha \geq 1$.
  In this case, $\tilde{\theta}_2 \geq \theta_1$, $SW^*_{CE-SUB} = \frac{1}{2}c\left(\frac{(2\alpha-1)c}{\alpha^2} - 2\right) + \frac{7}{8}$.
- Case 2-ii-a: $\frac{\alpha}{2} \leq c < \alpha$, $0 < \alpha < 1$.
  In this case, $\tilde{\theta}_2 < \theta_1$, $SW^*_{CE-SUB} = \frac{2\alpha^4-(1-\alpha)^2c^4+2(1-\alpha)^2\alpha c^3+\left(\alpha^2-\alpha^4\right)c^2+2(\alpha-2)\alpha^3 c}{2\alpha^2(\alpha+\alpha c-c)^2}$.
- Case 2-ii-b: $\frac{\alpha}{2} \leq c < \alpha$, $\alpha \geq 1$.
  In this case, $\tilde{\theta}_2 \geq \theta_1$, $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(\alpha-1)c+\alpha)}{\alpha^2}$. □

Proposition D.4. *Under S model, in the presence of adoption costs, the firm's optimal seeding ratio, pricing strategy, the corresponding profit, and ensuing social welfare are:*

	$0 < c < 2\alpha$		$2\alpha \leq c$
	Region A (described below)	Otherwise	
p^*_S	$\frac{2\alpha-(7\alpha+1)c+t}{16\alpha}$	p^*_{CE-PL}	-
k^*_S	$\frac{-8\alpha^2+2\alpha+(\alpha-1)c+t}{4(1-\alpha)(2\alpha-c)}$	0	-
π^*_S	$\frac{(6\alpha+3(\alpha-1)c-t)(2\alpha-(7\alpha+1)c+t)^2}{128(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c+t)}$	π^*_{CE-PL}	-
SW^*_S	$\tilde{SW}_{S,D}$	SW^*_{CE-PL}	-
Paid adoption	in both periods	same as *CE-PL*	none

where:

$$\begin{aligned}\tilde{SW}_{S,D} =& \frac{(2\alpha(1-c)+c)(2\alpha(1-4\alpha)+(\alpha-1)c+t)}{16(1-\alpha)\alpha^2} + \left((6\alpha+3(\alpha-1)c-t)\left(-16\alpha^3(c(16c+15)\right.\right.\\ &-18)+\alpha^2(c(c(86c+325)-228)+4(4t-3))+2\alpha(c(c(-46c+10t+15)-5t+6)-2t)\\ &+(2c-1)(c-t)(3c+t))\left(4\alpha^2(56\alpha-3)+(\alpha(29\alpha+38)-3)c^2-2c(2\alpha(\alpha(4\alpha+47)-3)\right.\\ &\left.\left.+(\alpha-1)t)+t^2-4\alpha(4\alpha+1)t\right)\right) / \left(2\left(4\alpha^2(64\alpha-3)-\left((\alpha-1)(43\alpha-3)c^2\right)\right.\right.\\ &\left.\left.+2c(2\alpha(\alpha(32\alpha-55)+3)-5\alpha t+t)+t^2-4\alpha t\right)^2(4(\alpha-1)(c-2\alpha))\right),\end{aligned}$$

and $t = (2\alpha + 17\alpha c - c)(\alpha(c + 2) - c)$. *Region A corresponds to parameters* α *and* c *satisfying:*

$$
\begin{aligned}
&0 \le c < c^{\ddagger}(\alpha) \quad , \textit{if } 0 < \alpha < \tfrac{1}{16}, \\
&\quad\quad \textit{and} \\
&\tfrac{1}{16} \le \alpha < \hat{\alpha}^{\dagger}(c).
\end{aligned}
$$

$c^{\ddagger}(\alpha)$ and $\hat{\alpha}^{\dagger}(c)$ are defined the proof below.

Proof. First, we point out that *CE-PL* is a particular case of *S* with seeding ratio set to zero. Throughout the proof, we will show that in certain regions *CE-PL* dominates *S* with non-zero seeding ratio - that is equivalent to saying that the optimal seeding ratio will be 0 in those regions (i.e., *S* defaults to *CE-PL*).

If $\alpha \ge 1$, seeding brings no benefit as any social learning calibrates perceived valuations downwards, and, as such, *S* defaults to *CE-PL*.

Thus, we are left to explore the non-trivial case of $0 < \alpha < 1$. It is straightforward that the firm can make profit iff $0 \le c < 2\alpha$. In the remaining part of the proof we focus on the scenario $0 \le c < 2\alpha$. We have two cases:

- Case 1: $0 < p < 2\alpha - c$.

  In this case, there are paying adopters in period 1 (potentially alongside seeded customers if $k > 0$). The marginal paying adopter in period 1 has type $\theta_1 = \frac{c+p}{2\alpha}$. The marginal seeded adopter in period 1 has type $\theta_{seed} = \frac{c}{2\alpha}$ (unlike in the baseline model, in the scenario with adoption cost not all seeded customers adopt).

  Thus, the total number of adopters in period 1 is $N_{1,total} = k\left(1 - \frac{c}{2\alpha}\right) + (1-k)\left(1 - \frac{c+p}{2\alpha}\right) = \frac{2\alpha - c - p(1-k)}{2\alpha}$. In period 2, the potential customers who have not adopted in period 1 update their prior beliefs via social learning as follows:

  $$a_2 = a_1 + N_{1,total}(1 - a_1) = \alpha + \frac{(1-\alpha)(2\alpha - c - p(1-k))}{2\alpha}.$$

  A customer of type θ who has not adopted in period 1 (via paying for license or through the seeding program) will adopt in period 2 iff $\theta_1 > \theta \ge \theta_2 = \frac{c+p}{\alpha + \frac{(1-\alpha)(2\alpha-c-p(1-k))}{2\alpha}}$. Comparing θ_1 and θ_2, we have:

  $$\theta_1 > \theta_2 \quad \Longleftrightarrow \quad p < \frac{-4\alpha^2 + 2\alpha + \alpha c - c}{(1-\alpha)(1-k)}.$$

  Comparing $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$ with 0, we have:

  $$\frac{-4\alpha^2 + 2\alpha + \alpha c - c}{(1-\alpha)(1-k)} > 0 \quad \Longleftrightarrow \quad 0 < \alpha < \frac{1}{2} \quad \text{and} \quad 0 \le c < \frac{2\alpha - 4\alpha^2}{1-\alpha} < 2\alpha.$$

  Comparing $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$ with $2\alpha - c$, we have:

  $$\frac{-4\alpha^2 + 2\alpha + \alpha c - c}{(1-\alpha)(1-k)} < 2\alpha - c \quad \Longleftrightarrow \quad 0 \le k < \frac{2\alpha^2}{(1-\alpha)(2\alpha - c)} < 1.$$

Since in this case we consider $p \in (0, 2\alpha - c)$, we have four sub-cases:

- Case 1-i: $0 < \alpha < \frac{1}{2}$, $0 \leq c < \frac{2\alpha - 4\alpha^2}{1-\alpha}$, $0 \leq k < \frac{2\alpha^2}{(1-\alpha)(2\alpha - c)}$.
  In this case, $0 < \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} < 2\alpha - c$. We have two sub-cases:
  * Case 1-i-a: $0 < p < \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$.
    In this case, $\theta_1 > \theta_2$. Customers with type $\theta \in [\theta_2, \theta_1)$, who have not been successfully seeded in period 1, adopt in period 2.
    The firm's profit maximization problem becomes:

$$\max_{0<p<\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}, 0\leq k<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} \pi_S$$
$$= \max_{0<p<\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}, 0\leq k<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} p(1-k)\left(1 - \frac{c+p}{\alpha + \frac{(\alpha-1)(-2\alpha+c-kp+p)}{2\alpha}}\right).$$

It can be shown that $\frac{\partial^2 \pi_S}{\partial p^2} < 0$. Thus, it is sufficient to solve FOC:

$$\frac{\partial \pi_S}{\partial p} = \frac{(k-1)\left(\alpha^2\left(c^2+2c(k+1)p+p(k(4-kp)+p+4)-4\right)+2\alpha(c(2-2kp)+(k-1)p(kp-2))-(c-kp+p)^2\right)}{(2\alpha+(\alpha-1)c+p(\alpha-\alpha k+k-1))^2} = 0.$$

Without constraints, the FOC yields two solutions:

$$p_{1,D,S} = \frac{2\alpha+(\alpha-1)c+\frac{\sqrt{2}\sqrt{\alpha(\alpha(c+2)-c)(\alpha+(\alpha-1)k+1)(2\alpha+(\alpha-1)ck)}}{\alpha+(\alpha-1)k+1}}{(1-\alpha)(1-k)},$$
$$p_{2,D,S} = \frac{2\alpha+(\alpha-1)c-\frac{\sqrt{2}\sqrt{\alpha(\alpha(c+2)-c)(\alpha+(\alpha-1)k+1)(2\alpha+(\alpha-1)ck)}}{\alpha+(\alpha-1)k+1}}{(1-\alpha)(1-k)}.$$

It can be shown that $p_{1,D,S} > \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$ and $p_{2,D,S} > 0$. Comparing $p_{2,D,S}$ with $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$, we have:

$$\begin{aligned}
& p_{2,D,S} < \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} \\
\iff \quad & 4\alpha^2 < \sqrt{2}\sqrt{\frac{\alpha(2\alpha+(\alpha-1)c)(2\alpha+(\alpha-1)ck)}{\alpha+(\alpha-1)k+1}} \\
\iff \quad & 8\alpha^4 < \frac{\alpha(2\alpha+(\alpha-1)c)(2\alpha+(\alpha-1)ck)}{\alpha+(\alpha-1)k+1} \\
\iff \quad & 8\alpha^4+8\alpha^3+2\alpha((1-\alpha)c-2\alpha)+k\left(8(\alpha-1)\alpha^3+(\alpha-1)c((1-\alpha)c-2\alpha)\right) < 0.
\end{aligned}$$

Without constraints, $8\alpha^4+8\alpha^3+2\alpha((1-\alpha)c-2\alpha)+k\left(8(\alpha-1)\alpha^3+(\alpha-1)c((1-\alpha)c-2\alpha)\right) = 0$ yields one solution:

$$k_{1,D,S} = \frac{2\alpha(2\alpha(2\alpha(\alpha+1)-1)-\alpha c+c)}{(1-\alpha)\left(8\alpha^3+c^2-\alpha c(c+2)\right)}.$$

Notice that:

$$8(\alpha-1)\alpha^3+(\alpha-1)c((1-\alpha)c-2\alpha)>0 \iff c(2\alpha+(\alpha-1)c)-8\alpha^3>0$$
$$k_{1,D,S}\geq 0 \iff \left(c(2\alpha+(\alpha-1)c)-8\alpha^3\right)(2\alpha(2\alpha(\alpha+1)-1)-\alpha c+c)\leq 0,$$
$$k_{1,D,S}<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \iff c(2\alpha+(\alpha-1)c)-8\alpha^3<0.$$

Then, we obtain four cases:

· Case 1-i-a-I: $c(2\alpha+(\alpha-1)c)-8\alpha^3\geq 0$.
In this case, $p_{2,D,S}<\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$. Thus, $p_S^*=p_{2,D,S}$. The profit maximization problem becomes:

$$\max_{0\leq k<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} \Big(-(\alpha-1)c(\alpha+(3\alpha-1)k+1)+2\sqrt{2}\sqrt{\alpha(2\alpha+(\alpha-1)c)(\alpha+(\alpha-1)k+1)(2\alpha+(\alpha-1)ck)}+$$
$$2\alpha(-\alpha(k+3)+k-1)\Big)/\left((\alpha-1)^2(k-1)\right).$$

It can be shown that $\frac{\partial\pi_S}{\partial k}<0$. Thus, $k_S^*=0$, S defaults to *CE-PL*.

· Case 1-i-a-II: $c(2\alpha+(\alpha-1)c)-8\alpha^3<0$, $2\alpha(2\alpha(\alpha+1)-1)+c-\alpha c\geq 0$, $0\leq k<k_{1,D,S}$.
In this case, $p_{2,D,S}>\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$, $p_S^*=\frac{2\alpha+\alpha c-2c}{-\alpha+\alpha k-k+2}$. The profit maximization problem becomes:

$$\max_{0\leq k<\frac{2\alpha(2\alpha(2\alpha(\alpha+1)-1)-\alpha c+c)}{(1-\alpha)\left(8\alpha^3+c^2-\alpha c(c+2)\right)}} \frac{(2\alpha-1)(k-1)(2\alpha+(\alpha-2)c)}{-\alpha+(\alpha-1)k+2}.$$

It can be shown that $\Delta_{PL,D}(\alpha,c)<0$ in this case, which corresponds to the second case under *CE-PL*. For any $k\in[0,k_{1,D,S})$, $\frac{(2\alpha-1)(k-1)(2\alpha+(\alpha-2)c)}{-\alpha+(\alpha-1)k+2}<\frac{(c-2\alpha)^2}{8\alpha}=\pi^*_{CE-PL}$. Therefore, this case is sub-optimal, as it is dominated by not seeding anymore.

· Case 1-i-a-III: $c(2\alpha+(\alpha-1)c)-8\alpha^3<0$, $2\alpha(2\alpha(\alpha+1)-1)+c-\alpha c\geq 0$, $k_{1,D,S}\leq k<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$.
In this case, $p_{2,D,S}<\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$. Thus, $p_S^*=p_{2,D,S}$. It can be shown $\frac{\partial\pi}{\partial k}<0$ as well. Therefore, $k_S^*=k_{1,D,S}$. It can be shown that $\Delta_{PL,D}(\alpha,c)<0$ in this case, which corresponds to the second case under *CE-PL*. For any $k\in\left[k_{1,D,S},\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}\right)$, $\pi_S<\frac{(c-2\alpha)^2}{8\alpha}=\pi^*_{CE-PL}$. Therefore, this case is sub-optimal, as it is dominated by not seeding anymore.

· Case 1-i-a-IV: $c(2\alpha+(\alpha-1)c)-8\alpha^3<0$, $2\alpha(2\alpha(\alpha+1)-1)+c-\alpha c<0$.
In this case, $p_{2,D,S}<\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$. Thus, $p_S^*=p_{2,D,S}$. Same as case 1-i-a-I, $\frac{\partial\pi_S}{\partial k}<0$. Thus, $k_S^*=0$, S defaults to *CE-PL*.

Thus, under case 1-i-a, S either defaults to *CE-PL* or is strictly dominated by *CE-PL*.

* Case 1-i-b: $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}\leq p<2\alpha-c$.
In this case, $\theta_1\leq\theta_2$. There are no new adopters in period 2. The firm's profit

maximization problem becomes:

$$\begin{aligned}
&\max_{\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}\le p<2\alpha-c,0\le k<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} \pi_S \\
=&\max_{\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}\le p<2\alpha-c,0\le k<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} p(1-k)\left(1-\frac{c+p}{2\alpha}\right).
\end{aligned}$$

It trivially follows that $k_S^* = 0$. S defaults to *CE-PL*.

- Case 1-ii: $0 < \alpha < \frac{1}{2}$, $0 \le c < \frac{2\alpha-4\alpha^2}{1-\alpha}$, $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \le k < 1$.
  In this case, $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} \ge 2\alpha - c$. $\theta_2 < \theta_1$. Customers with type $\theta \in [\theta_2, \theta_1)$, who have not been seeded in period 1, adopt in period 2. The firm's profit maximization problem becomes:

$$\begin{aligned}
&\max_{0<p<2\alpha-c,\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}\le k<1} \pi_S \\
=&\max_{0<p<2\alpha-c,\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}\le k<1} p(1-k)\left(1-\frac{c+p}{\alpha+\frac{(\alpha-1)(-2\alpha+c-kp+p)}{2\alpha}}\right).
\end{aligned}$$

  Similarly to case 1-i-a, it can be shown that $p_{1,D,S} > 2\alpha - c$ and $p_{2,D,S} > 0$. Comparing $p_{2,D,S}$ with $2\alpha - c$, we have:

$$\begin{aligned}
&p_{2,D,S} < 2\alpha - c \\
\iff\ & (\alpha-1)ck + 2\alpha(\alpha-\alpha k+k) < \sqrt{2}\sqrt{\frac{\alpha(2\alpha+(\alpha-1)c)(2\alpha+(\alpha-1)ck)}{\alpha+(\alpha-1)k+1}} \\
\iff\ & ((\alpha-1)ck + 2\alpha(\alpha-\alpha k+k))^2 < \frac{2\alpha(2\alpha+(\alpha-1)c)(2\alpha+(\alpha-1)ck)}{\alpha+(\alpha-1)k+1} \\
\iff\ & (1-\alpha)^2(c-2\alpha)^2k^2 + 2(1-\alpha)(2\alpha-c)\alpha ck + 4\alpha^2(c-\alpha(\alpha+2)) < 0.
\end{aligned}$$

  Without constraints, $(1-\alpha)^2(c-2\alpha)^2k^2 + 2(1-\alpha)(2\alpha-c)\alpha ck + 4\alpha^2(c-\alpha(\alpha+2)) = 0$ yields two solutions:

$$\begin{aligned}
k_{2,D,S} &= \frac{-\alpha c + \sqrt{\alpha^2(4\alpha(\alpha+2)+(c-4)c)}}{(1-\alpha)(2\alpha-c)} \\
k_{3,D,S} &= \frac{-\alpha c - \sqrt{\alpha^2(4\alpha(\alpha+2)+(c-4)c)}}{(1-\alpha)(2\alpha-c)}.
\end{aligned}$$

  It can be shown that $k_{3,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$ and $k_{2,D,S} > \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Comparing $k_{2,D,S}$ with 1, we obtain three cases:

  * Case 1-ii-a: $\alpha\left(2\alpha + \sqrt{4\alpha(\alpha+2)+(c-4)c} - 2\right) + c \ge 2\alpha c$.
    In this case, $k_{2,D,S} \ge 1$, i.e., $p_{2,D,S} < 2\alpha - c$. Thus, $p_S^* = p_{2,D,S}$. The profit

maximization problem becomes:

$$\max_{\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}\le k<1}\Big(-(\alpha-1)c(\alpha+(3\alpha-1)k+1)+2\sqrt{2}\sqrt{\alpha(2\alpha+(\alpha-1)c)(\alpha+(\alpha-1)k+1)(2\alpha+(\alpha-1)ck)}+$$
$$2\alpha(-\alpha(k+3)+k-1)\Big)\,/\,\big((\alpha-1)^2(k-1)\big)\,.$$

It can be shown that $\frac{\partial \pi_S}{\partial k}<0$. Thus, $k_S^*=\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Therefore, this case is weakly dominated by case 1-i. Thus, this case is strictly dominated by *CE-PL*.

- ∗ Case 1-ii-b: $\alpha\left(2\alpha+\sqrt{4\alpha(\alpha+2)+(c-4)c}-2\right)+c<2\alpha c$, $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}\le k<k_{2,D,S}$.
  In this case, $p_{2,D,S}<2\alpha-c$. Thus, $p_S^*=p_{2,D,S}$. Similarly as case 1-ii-a, we get $\frac{\partial \pi_S}{\partial k}<0$. Thus, $k_S^*=\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Therefore, this case is also weakly dominated and is strictly dominated by *CE-PL*.
- ∗ Case 1-ii-c: $\alpha\left(2\alpha+\sqrt{4\alpha(\alpha+2)+(c-4)c}-2\right)+c<2\alpha c$, $k_{2,D,S}\le k<1$.
  In this case, $p_{2,D,S}\ge 2\alpha-c$. We can see that, for any k in this region, $\pi_S(p)$ is strictly increasing in p and the profit in this case is strictly dominated by the profit under Case 2.

– Case 1-iii: $0<\alpha<\frac{1}{2}$, $\frac{2\alpha-4\alpha^2}{1-\alpha}\le c<2\alpha$.
  In this case, $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}\le 0$. $\theta_2\ge\theta_1$, the profit maximization problem becomes:

$$\max_{0<p<2\alpha-c,0\le k\le 1}\pi_S=\max_{0<p<2\alpha-c,0\le k\le 1}p(1-k)\left(1-\frac{c+p}{2\alpha}\right).$$

  It trivially follows that $k_S^*=0$. S defaults to *CE-PL*.

– Case 1-iv: $\frac{1}{2}\le\alpha<1$.
  In this case, $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}\le 0$. $\theta_2\ge\theta_1$, the profit maximization problem becomes:

$$\max_{0<p<2\alpha-c,0\le k\le 1}\pi_S=\max_{0<p<2\alpha-c,0\le k\le 1}p(1-k)\left(1-\frac{c+p}{2\alpha}\right).$$

  It trivially follows that $k_S^*=0$. S defaults to *CE-PL*.

- Case 2: $p\ge 2\alpha-c$.
  In this case, there are only seeded consumers in period 1 (i.e., no unseeded customer is willing to pay for the product based on priors). Hence, $N_{1,total}=k(1-\frac{c}{2\alpha})$. At the beginning of period 2, the un-seeded customers update their priors to:

$$a_2=a_1+N_{1,total}(1-a_1)=\alpha+(1-\alpha)k\left(1-\frac{c}{2\alpha}\right).$$

  The marginal paying customer in period 2 has type $\theta_2=\frac{c+p}{\alpha+(1-\alpha)k\left(1-\frac{c}{2\alpha}\right)}$. Comparing θ_2 with 1, we obtain:

$$\theta_2<1\quad\iff\quad p<\alpha-\alpha k+k+\frac{1}{2}c\left(-\frac{k}{\alpha}+k-2\right).$$

Comparing $\alpha - \alpha k + k + \frac{1}{2}c\left(-\frac{k}{\alpha} + k - 2\right)$ with $2\alpha - c$, we have:

$$\alpha - \alpha k + k + \frac{1}{2}c\left(-\frac{k}{\alpha} + k - 2\right) \geq 2\alpha - c \quad \Longleftrightarrow \quad k \geq \frac{2\alpha^2}{(1-\alpha)(2\alpha - c)} > 0.$$

Comparing $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$ with 1, we have:

$$\frac{2\alpha^2}{(1-\alpha)(2\alpha - c)} < 1 \quad \Longleftrightarrow \quad 0 < \alpha < \frac{1}{2} \quad \text{and} \quad 0 < c < \frac{2\alpha(1-2\alpha)}{1-\alpha}.$$

Thus, we obtain that:

$$\theta_2 < 1 \quad \Longleftrightarrow 0 < \alpha < \frac{1}{2}, 0 < c < \frac{2\alpha(1-2\alpha)}{1-\alpha}, \frac{2\alpha^2}{(1-\alpha)(2\alpha - c)} \leq k < 1,$$
$$\text{and } 2\alpha - c \leq p < \alpha - \alpha k + k + \frac{1}{2}c\left(-\frac{k}{\alpha} + k - 2\right).$$

Otherwise, $\theta_2 \geq 1$. There are no paying adopters in period 2, i.e., the firm does not make any profit.

When $\theta_2 < 1$, the firm's profit maximization problem becomes:

$$\max_{2\alpha - c \leq p < \alpha - \alpha k + k + \frac{1}{2}c\left(-\frac{k}{\alpha}+k-2\right), \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < 1} \pi_S$$
$$= \max_{2\alpha - c \leq p < \alpha - \alpha k + k + \frac{1}{2}c\left(-\frac{k}{\alpha}+k-2\right), \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < 1} p(1-k)\left(1 - \frac{c+p}{\alpha + (1-\alpha)k\left(1 - \frac{c}{2\alpha}\right)}\right).$$

Since it is quadratic in p, it is sufficient to use FOC. Taking the first order derivative of the profit w.r.t. p, we get:

$$\frac{\partial \pi_S}{\partial p} = \frac{2\alpha(k-1)(c+2p)}{(\alpha-1)ck + 2\alpha(\alpha - \alpha k + k)} - k + 1.$$

Without constraints, the FOC yields one solution:

$$p_{3,D,S} = \frac{\alpha c(k-2) - ck + 2\alpha(\alpha - \alpha k + k)}{4\alpha}.$$

It can be shown that $p_{3,D,S} < \alpha + \frac{1}{2}c\left(-\frac{k}{\alpha} + k - 2\right) - \alpha k + k$. Comparing $p_{3,D,S}$ with $2\alpha - c$, we obtain:

$$p_{3,D,S} \geq 2\alpha - c \quad \Longleftrightarrow \quad k \geq \frac{2\alpha(3\alpha - c)}{(1-\alpha)(2\alpha - c)}.$$

It can be shown that $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)} > \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Comparing $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$ with 1, we obtain:

$$\frac{2\alpha(3\alpha - c)}{(1-\alpha)(2\alpha - c)} < 1 \quad \Longleftrightarrow \quad \left(0 < \alpha < \frac{1}{3} \text{ and } c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}\right) \text{ or } \left(\frac{1}{3} < \alpha < \frac{1}{2} \text{ and } c > \frac{2\alpha(1-4\alpha)}{1-3\alpha}\right).$$

Comparing $\frac{2\alpha(1-4\alpha)}{1-3\alpha}$ with 0 and $\frac{2\alpha(1-2\alpha)}{1-\alpha}$, we obtain three cases:

- Case 2-i: $0 < \alpha < \frac{1}{4}$.
  In this case, $0 < \frac{2\alpha(1-4\alpha)}{1-3\alpha} < \frac{2\alpha(1-2\alpha)}{1-\alpha}$. We obtain three cases:
  * Case 2-i-a: $0 \leq c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$, $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$.
    In this case, $p_{3,D,S} < 2\alpha - c$. Thus, $p_S^* = 2\alpha - c$. The profit maximization problem becomes:

$$\begin{aligned}&\max_{\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}} \pi_S \\ =& \max_{\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}} (1-k)(2\alpha-c)\left(1 - \frac{2\alpha}{\alpha + \frac{(\alpha-1)k(c-2\alpha)}{2\alpha}}\right).\end{aligned}$$

    It can be shown that $\frac{\partial^2 \pi_S}{\partial k^2} < 0$. Hence, FOC is sufficient to determine the optimal seeding ratio. We have:

$$\frac{\partial \pi_S}{\partial k} = (2\alpha - c)\left(\frac{4\alpha^2(2\alpha + (\alpha-1)c)}{((\alpha-1)ck + 2\alpha(\alpha - \alpha k + k))^2} - 1\right) = 0.$$

    Without constraints, FOC yields two solutions:

$$\begin{aligned}k_{4,D,S} &= \frac{-2\alpha^2 + 2\alpha\sqrt{2\alpha + (\alpha-1)c}}{(1-\alpha)(2\alpha-c)}, \\ k_{5,D,S} &= \frac{-2\alpha^2 - 2\alpha\sqrt{(2\alpha + (\alpha-1)c)}}{(1-\alpha)(2\alpha-c)}.\end{aligned}$$

    It can be shown that $k_{5,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$ and $k_{4,D,S} > \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Comparing $k_{4,D,S}$ with 1, we obtain two sub cases:
    · Case 2-i-a-I: $\sqrt{\alpha(c+2) - c} + c < 4\alpha$.
      In this case, $k_{4,D,S} < 1$, $p_S^* = 2\alpha - c$, $k_S^* = k_{4,D,S}$, $\pi_S^* = \frac{-\alpha c + 2\alpha\left(-2\alpha + 2\sqrt{\alpha(c+2)-c} - 1\right) + c}{\alpha - 1}$. It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.
    · Case 2-i-a-II: $\sqrt{\alpha(c+2) - c} + c \geq 4\alpha$.
      In this case, $k_{4,D,S} \geq 1$, $p_S^* = 2\alpha - c$, $k_S^* = 1$, $\pi_S^* = 0$. Thus, it is dominated by *CE-PL*.
  * Case 2-i-b: $0 \leq c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$, $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)} \leq k < 1$.
    In this case, $p_{3,D,S} \geq 2\alpha - c$. Thus, $p_S^* = p_{3,D,S}$. The firm's profit maximization problem becomes:

$$\begin{aligned}&\max_{\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)} \leq k \leq 1} \pi_S \\ =& \max_{\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)} \leq k \leq 1} \frac{(k-1)(\alpha c(k-2) - ck + 2\alpha(\alpha - \alpha k + k))^2}{8\alpha\left(-\alpha(c+2)k + ck + 2\alpha^2(k-1)\right)}.\end{aligned}$$

We differentiate π_S w.r.t. k:

$$\begin{aligned}\frac{\partial \pi_S}{\partial k} = &\big((c(2\alpha-\alpha k+k)+2\alpha((\alpha-1)k-\alpha)\big)\big((\alpha-1)c^2((\alpha-1)k(2k-1)-2\alpha)- \\ &2\alpha c\left(\alpha^2+\alpha+4(\alpha-1)^2k^2+((7-5\alpha)\alpha-2)k\right)+4\alpha^2((\alpha-1)k-\alpha)(-2\alpha+ \\ &2(\alpha-1)k+1))\big) / \left(8\alpha((\alpha-1)ck+2\alpha(\alpha-\alpha k+k))^2\right).\end{aligned}$$

It can be shown that: $\frac{c(2\alpha-\alpha k+k)+2\alpha((\alpha-1)k-\alpha)}{8\alpha((\alpha-1)ck+2\alpha(\alpha-\alpha k+k))^2} < 0$. Denote:

$$\begin{aligned}G_{D,S}(k) \triangleq &-\big((\alpha-1)c^2((\alpha-1)k(2k-1)-2\alpha)-2\alpha c\left(\alpha^2+\alpha+4(\alpha-1)^2k^2\right. \\ &\left.+((7-5\alpha)\alpha-2)k\right)+4\alpha^2((\alpha-1)k-\alpha)(-2\alpha+2(\alpha-1)k+1)\big) \\ = &-2(1-\alpha)^2(2\alpha-c)^2k^2 \\ &+(\alpha-1)\left(4\alpha^2(4\alpha-1)+(\alpha-1)c^2+2\alpha(2-5\alpha)c\right)k \\ &+2\alpha\left(2\alpha^2(1-2\alpha)+(\alpha-1)c^2+\alpha(\alpha+1)c\right).\end{aligned}$$

It is straightforward that $G_{D,S}(k)$ is concave. Without constraints, $G_{D,S}(k) = 0$ yields two solutions:

$$\begin{aligned}k_{6,D,S} &= \frac{-8\alpha^2+2\alpha+(\alpha-1)c+\sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}}{4(1-\alpha)(2\alpha-c)}, \\ k_{7,D,S} &= \frac{-8\alpha^2+2\alpha+(\alpha-1)c-\sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}}{4(1-\alpha)(2\alpha-c)}.\end{aligned}$$

It can be shown that $k_{7,D,S} < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$ and $k_{6,D,S} < 1$. Comparing $k_{6,D,S}$ with $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$, we obtain two sub-cases:

· Case 2-i-b-I: $\alpha(9c+2)+\sqrt{(2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)} > 32\alpha^2+c$.
In this case, $k_{6,D,S} > \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$. Denote $t = (2\alpha+17\alpha c-c)(\alpha(c+2)-c)$, We can further get:
$p_S^* = \frac{2\alpha^2-2\alpha c+\alpha ck-ck-2\alpha^2k+2\alpha k}{4\alpha} = \frac{2\alpha-(7\alpha+1)c+t}{16\alpha}$, $k_S^* = k_{6,D,S} = \frac{-8\alpha^2+2\alpha+(\alpha-1)c+t}{4(1-\alpha)(2\alpha-c)}$,
$\pi_S^* = \frac{(6\alpha+3(\alpha-1)c-t)(2\alpha-(7\alpha+1)c+t)^2}{128(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c+t)}$,

$$\begin{aligned}SW_S^* = &\frac{(2\alpha(1-c)+c)(2\alpha(1-4\alpha)+(\alpha-1)c+t)}{16(1-\alpha)\alpha^2} + \big((6\alpha+3(\alpha-1)c-t)\left(-16\alpha^3(c(16c+15)\right. \\ &-18)+\alpha^2(c(c(86c+325)-228)+4(4t-3))+2\alpha(c(c(-46c+10t+15)-5t+6)-2t) \\ &+(2c-1)(c-t)(3c+t)\big)\left(4\alpha^2(56\alpha-3)+(\alpha(29\alpha+38)-3)c^2-2c(2\alpha(\alpha(4\alpha+47)-3)\right. \\ &+(\alpha-1)t)+t^2-4\alpha(4\alpha+1)t\big)\big) / \big(2\left(4\alpha^2(64\alpha-3)-((\alpha-1)(43\alpha-3)c^2)\right. \\ &\left.+2c(2\alpha(\alpha(32\alpha-55)+3)-5\alpha t+t)+t^2-4\alpha t\right)^2(4(\alpha-1)(c-2\alpha))\big).\end{aligned}$$

· Case 2-i-b-II: $\alpha(9c+2)+\sqrt{(2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)} \leq 32\alpha^2+c$.
In this case, $k_{6,D,S} \leq \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$. Thus, $k_S^* = \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$, $\pi_S^* = \frac{(2\alpha-c)(2\alpha(4\alpha-1)-3\alpha c+c)}{(\alpha-1)(4\alpha-c)}$.
It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.

∗ Case 2-i-c: $\frac{2\alpha(1-4\alpha)}{1-3\alpha} \leq c < \frac{2\alpha(1-2\alpha)}{1-\alpha}$.

In this case, $p_{3,D,S} < 2\alpha - c$. Thus, $p_S^* = 2\alpha - c$. Following the same step in case 2-i-a, we obtain that $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} < k_{4,D,S} < 1$ and $k_{5,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Thus, $k_S^* = k_{4,D,S}$, $\pi_S^* = \frac{-\alpha c+2\alpha\left(-2\alpha+2\sqrt{\alpha(c+2)-c}-1\right)+c}{\alpha-1}$. It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.

- Case 2-ii: $\frac{1}{4} \leq \alpha < \frac{1}{3}$.
  In this case, $\frac{2\alpha(1-4\alpha)}{1-3\alpha} \leq 0$. $p_{3,D,S} \leq 2\alpha - c$. Thus, $p_S^* = 2\alpha - c$. Following the same step in case 2-i-a, we obtain that $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} < k_{4,D,S} < 1$ and $k_{5,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Thus, $k_S^* = k_{4,D,S}$, $\pi_S^* = \frac{-\alpha c+2\alpha\left(-2\alpha+2\sqrt{\alpha(c+2)-c}-1\right)+c}{\alpha-1}$. It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.
- Case 2-iii: $\frac{1}{3} \leq \alpha < \frac{1}{2}$.
  In this case, $\frac{2\alpha(1-4\alpha)}{1-3\alpha} \geq \frac{2\alpha(1-2\alpha)}{1-\alpha}$. Thus, $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)} \geq 1$. $p_{3,D,S} < 2\alpha - c$. Thus, $p_S^* = 2\alpha - c$. Following the same step in case 2-i-a, we obtain that $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} < k_{4,D,S} < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$ and $k_{5,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Thus, $k_S^* = k_{4,D,S}$, $\pi_S^* = \frac{-\alpha c+2\alpha\left(-2\alpha+2\sqrt{\alpha(c+2)-c}-1\right)+c}{\alpha-1}$. It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.

In summary, only under case 2-i-b-I, S can be optimal. We further explore the boundary between S and *CE-PL*. Recall that the condition for case 2-i-b-I is: $0 \leq c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$, and $\alpha(9c+2) + \sqrt{(2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)} > 32\alpha^2 + c$. This region is only relevant to case (a) and case (b) under *CE-PL*. The last inequality can be rewrite as:

$$\sqrt{(2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)} > 32\alpha^2 + c - \alpha(9c+2). \tag{D.3}$$

We first check whether the R.H.S. is positive. Denote $H_{S,1}(\alpha, c) \triangleq 32\alpha^2 + c - \alpha(9c+2) = 2\alpha(16\alpha - 1) + (1-9\alpha)c$. We obtain two cases (we reorganize the case number to avoid it goes too deep):

- Case 1: $0 < \alpha < \frac{1}{9}$.
  In this case, $H_{S,1}(\alpha, c)$ is increasing in c, $32\alpha^2 + c - \alpha(9c+2) > 0$ is equivalent to $c > \frac{2\alpha(1-16\alpha)}{1-9\alpha}$. It can be shown that $\frac{2\alpha(1-16\alpha)}{1-9\alpha} < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. Comparing $\frac{2\alpha(1-16\alpha)}{1-9\alpha}$ with 0, we obtain two sub cases:
  - Case 1-i: $0 < \alpha < \frac{1}{16}$.
    In this case, $0 < \frac{2\alpha(1-16\alpha)}{1-9\alpha} < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. We obtain two sub cases:
    * Case 1-i-a: $0 \leq c < \frac{2\alpha(1-16\alpha)}{1-9\alpha}$.
      In this case, $32\alpha^2 + c - \alpha(9c+2) < 0$, the inequality D.3 is always satisfied. Recall that for *CE-PL*, the boundary between two cases is $c^\dagger(\alpha)$, where $c^\dagger(\alpha)$ is the unique solution to the equation $\Phi_{PL,D}(\alpha, c) = 0$ and $\Phi_{PL,D}(\alpha, c)$ is decreasing in c. It can be shown that $\Phi_{PL,D}(\alpha, c)\Big|_{c=\frac{2\alpha(1-16\alpha)}{1-9\alpha}} > 0$. Thus, $\frac{2\alpha(1-16\alpha)}{1-9\alpha} < c^\dagger(\alpha)$. This case falls into the region of first case under *CE-PL*.

Next, we compare the optimal profit between S and *CE-PL*. We first simplify the optimal profit under S as (move the square root to the numerator):

$$\pi_S^* = \Big(\sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}\left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)c^2+4\alpha(9\alpha-1)c\right)$$
$$-\left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3+2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big)$$
$$/\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right).$$

It can be shown that under this case, $\pi_S^* > \pi_{CE-PL}^*$. Thus, S dominates *CE-PL*.

* Case 1-i-b: $\frac{2\alpha(1-16\alpha)}{1-9\alpha} \le c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$.
  In this case, $32\alpha^2 + c - \alpha(9c+2) \ge 0$. We take square both sides of the inequality D.3. After the simplification the inequality is equivalent to:

$$c^2 + (1-9\alpha)c + 2\alpha(8\alpha-1) < 0,$$

  which is equivalent to:

$$\frac{1}{2}\left(-\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) < c < \frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right).$$

  It can be shown that:

$$\frac{1}{2}\left(-\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) < \frac{2\alpha(1-16\alpha)}{1-9\alpha}$$
$$< \frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) < \frac{2\alpha(1-4\alpha)}{1-3\alpha}.$$

  Thus, the inequality D.3 is equivalent to: $\frac{2\alpha(1-16\alpha)}{1-9\alpha} \le c < \frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)$.
  Then we compare S with *CE-PL*. We first check the relationship between $\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)$ and $c^\dagger(\alpha)$.
  It can be shown that $\Phi_{PL,D}(\alpha,c)\Big|_{c=\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)} > 0$ is equivalent to $\frac{1}{17} < \alpha < \frac{1}{16}$. We further get two sub cases:
  - Case 1-i-b-I: $0 < \alpha < \frac{1}{17}$.
    In this case, we have $\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) > c^\dagger(\alpha)$. Thus, we consider two regions:
    Region 1: $\frac{2\alpha(1-16\alpha)}{1-9\alpha} \le c < c^\dagger(\alpha)$**.**
    In this region, denote the profit difference between S and *CE-PL* as:

$$H_{S,2} \triangleq \Big(\sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}\left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)c^2+4\alpha(9\alpha-1)c\right)$$
$$-\left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3+2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big)$$
$$/\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right)$$
$$-\frac{2\alpha+\alpha^2(c+6)-4\sqrt{\alpha^2(\alpha+1)(2\alpha+(\alpha-1)c)}-c}{(1-\alpha)^2}.$$

    Thus, the boundary between S and *CE-PL* satisfies: $H_{S,2} = 0$. We simplify the equation $H_{S,2} = 0$ by getting rid of the fraction and square root. We finally get

$H_{S,2} = 0$ is equivalent to $H_{S,3} = 0$, where $H_{S,3}$ is defined as:

$$\begin{aligned} H_{S,3} \triangleq & 4(\alpha+1)(2\alpha-c)^2(2\alpha+(\alpha-1)c)\left(c^2-128\alpha^4(c+6)+5\alpha^3(c(27c+92)-52)\right. \\ & \left.+\alpha^2(c(184-109c)+4)-\alpha c(27c+4)\right)^2-\left(8\alpha^3(\alpha(\alpha(544\alpha+451)+30)\right. \\ & -1)+2(\alpha-1)^2(\alpha+1)(26\alpha-1)c^4-(\alpha-1)(\alpha(\alpha(5\alpha(27\alpha-134)-604) \\ & -14)+1)c^3+2\alpha(\alpha(\alpha(\alpha(\alpha(32\alpha-1147)+760)+1310)+72)-3)c^2 \\ & \left.+4\alpha^2(\alpha(\alpha(\alpha(448\alpha-1137)-1279)-83)+3)c\right)^2. \end{aligned}$$

We can obtain that:

$$\begin{aligned} \frac{\partial H_{S,3}(\alpha,c)}{\partial\alpha} = & 2(1-\alpha)\left(64(\alpha-1)\alpha^5(\alpha(\alpha(32\alpha(192\alpha-287)+3653)-232)+3)\right. \\ & +4(\alpha-1)^2(\alpha+1)(26\alpha-1)(\alpha(104\alpha+23)-27)c^8 \\ & -4(\alpha-1)(\alpha(\alpha(\alpha(\alpha(5\alpha(3159\alpha+1238)-36154)-7416)+12637) \\ & -658)+6)c^7+(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(124405\alpha-24086)-663362) \\ & +406518)+503028)-359002)+30266)-1414)+31)c^6 \\ & -2(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(3\alpha(10\alpha(1584\alpha+1693)-197561)+329621) \\ & +774877)-651119)+108631)-10665)+375)-3)c^5 \\ & +4\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(96\alpha(64\alpha+603)-216517)+132899)+637875) \\ & -745325)+236785)-33335)+1425)-15)c^4-16\alpha^2(\alpha(\alpha(\alpha(\alpha(2\alpha \\ & (16\alpha(384\alpha+395)+13645)+142805)-295747)+151714)-26660) \\ & +1285)-15)c^3+16\alpha^3(\alpha(\alpha(\alpha(3\alpha(\alpha(64\alpha(192\alpha+269)+25401) \\ & -108576)+221380)-46398)+2445)-30)c^2-32\alpha^4(\alpha(\alpha(\alpha(2\alpha(48 \\ & \left.\alpha(256\alpha+223)-54221)+84329)-20999)+1191)-15)c\right). \end{aligned}$$

$$\begin{aligned} \frac{\partial H_{S,3}(\alpha,c)}{\partial c} = & 2(1-\alpha)^2(32\alpha^5(\alpha(\alpha(\alpha(32\alpha(64\alpha+119)-7417)+2777)-195)+3) \\ & -16(-26\alpha^3+\alpha^2+26\alpha-1)^2c^7+28(\alpha-1)\alpha(\alpha+1)(\alpha(\alpha(5\alpha(351\alpha \\ & +67)-4007)+323)-6)c^6-3(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(24881\alpha+16764) \\ & -151172)-13428)+156486)-18412)+1324)-60)+1)c^5 \\ & +10\alpha(\alpha(\alpha(\alpha(\alpha(\alpha\left(4320\alpha^2+8967\alpha-57883\right)-9445)+102601) \\ & -23019)+3311)-183)+3)c^4-8\alpha^2(\alpha(\alpha(\alpha(\alpha(\alpha(64\alpha(16\alpha+179) \\ & -32993)+206)+159649)-76108)+15505)-930)+15)c^3 \\ & +48\alpha^3(\alpha(\alpha(\alpha(\alpha(32\alpha(32\alpha+65)+4601)+19957)-19506)+4954) \\ & -315)+5)c^2-16\alpha^4(\alpha(\alpha(\alpha(3\alpha(128\alpha(16\alpha+39)+9573)-46856) \\ & +14346)-960)+15)c)<0. \end{aligned}$$

As it turns out, in this range of the parameter space, $\frac{\partial H_{S,3}(\alpha,c)}{\partial\alpha}$ changes signs. As such, it is not possible to characterize the threshold between S and *CE-PL*

as a function of c (there exist values of c for which increasing α leads to multiple crossings between optimality regions for S and *CE-PL*).
Nevertheless, moving horizontally, given that $\frac{\partial H_{S,3}(\alpha,c)}{\partial c} < 0$, a threshold (crossing) boundary between optimality regions for *CE-PL* and S, within this particular region of the parameter space, is unique for every α, *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region.

We look at two particular cases for this region:
(1) First, we consider points on the boundary $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$. It can be shown that $H_{S,3}(\alpha, c)\Big|_{c=\frac{2\alpha(1-16\alpha)}{1-9\alpha}} > 0$. Thus, S dominates *CE-PL* on $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$.
(2) First, we consider points on the boundary $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$. It can be shown that $H_{S,3}(\alpha, c)\Big|_{c=\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)} < 0$. Thus, *CE-PL* dominates S on $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$.

Therefore, in Region 1, as we increase c, there can be at most one crossing point between optimality regions for S and *CE-PL*, then there exists a unique boundary, which we define as $c_a(\alpha)$, which separates the optimality regions for S and *CE-PL*. It satisfies:

$$H_{S,3}(\alpha, c_1(\alpha)) = 0.$$

Region 2: $c^{\dagger}(\alpha) \leq c < \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$**.**

In this region, denote the profit difference between S and *CE-PL* as:

$$\begin{aligned} H_{S,4} \triangleq & \Big(\sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)}\left(4\alpha^2 + \left(17\alpha^2 - 18\alpha + 1\right)c^2 + 4\alpha(9\alpha - 1)c\right) \\ & - \left(-8\alpha^3 + \left(71\alpha^3 - 109\alpha^2 + 37\alpha + 1\right)c^3 + 2\alpha\left(109\alpha^2 - 74\alpha - 3\right)c^2 + 4\alpha^2(37\alpha + 3)c\right)\Big) \\ & / \left(64(1-\alpha)\alpha(2\alpha - c)(2\alpha + (\alpha - 1)c)\right) \\ & - \frac{(c - 2\alpha)^2}{8\alpha}. \end{aligned}$$

Thus, the boundary between S and *CE-PL* satisfies: $H_{S,4} = 0$. We simplify the equation $H_{S,4} = 0$ by getting rid of the fraction and square root. We finally get $H_{S,5} = 0$ is equivalent to $H_{S,5} = 0$, where $H_{S,5}$ is defined as:

$$\begin{aligned} H_{S,5} \triangleq & (2\alpha + (\alpha - 1)c)(2\alpha + (17\alpha - 1)c)^3 - (-4\alpha^2(16(\alpha - 1)\alpha + 1) + 8(\alpha - 1)c^3 \\ & + (\alpha(23\alpha + 10) - 1)c^2 + 4\alpha(\alpha(24\alpha - 5) + 1)c)^2. \end{aligned}$$

We can obtain that:

$$\begin{aligned}\frac{\partial H_{S,5}(\alpha,c)}{\partial\alpha} =& -4(-4\alpha^2(16(\alpha-1)\alpha+1)+8(\alpha-1)c^3+(\alpha(23\alpha+10)-1)c^2 \\ &+4\alpha(\alpha(24\alpha-5)+1)c)(-4\alpha(8\alpha(4\alpha-3)+1)+4c^3+(23\alpha+5)c^2 \\ &+2(2\alpha(36\alpha-5)+1)c)+(c+2)(2\alpha+(17\alpha-1)c)^3 \\ &+3(17c+2)(2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)^2>0.\end{aligned}$$

$$\begin{aligned}\frac{\partial H_{S,5}(\alpha,c)}{\partial c} =& 16(16\alpha^4(\alpha(6\alpha-5)(8\alpha-3)-1)-24(\alpha-1)^2c^5 \\ &-5(\alpha-1)(\alpha(23\alpha+10)-1)c^4+8\alpha(\alpha(\alpha(89\alpha-137)+15)+1)c^3 \\ &+12\alpha^2(\alpha((133-53\alpha)\alpha-34)+2)c^2 \\ &+16\alpha^3(\alpha(\alpha(17-49\alpha)+10)-2)c)<0.\end{aligned}$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-PL* and S within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists.*

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular functions for this region, namely $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$ and $c = \frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)$ and examine the sign of $H_{S,5}(\alpha,c)$ along these boundaries.

(1) On the boundary $c = \frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)$, we obtain: $H_{S,5}(\alpha,c)\Big|_{c=\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)} < 0$. Thus, *CE-PL* dominates S on $c = \frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)$.

(2) On the boundary $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$, we obtain: $H_{S,5}(\alpha,c)\Big|_{c=\frac{2\alpha(1-16\alpha)}{1-9\alpha}} > 0$. Thus, S dominates *CE-PL* on $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$.

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{c_b(\alpha)}$, which separates the optimality regions for *CE-PL* and S. It satisfies:

$$H_{S,5}(\alpha,c_b(\alpha))=0.$$

Also, it is straightforward that $\frac{\partial c_b(\alpha)}{\partial\alpha} = -\frac{\frac{\partial H_{S,5}(\alpha,c)}{\partial\alpha}}{\frac{\partial H_{S,5}(\alpha,c)}{\partial c}} > 0$. Hence, $c_b(\alpha)$ is increasing in α.

It can be shown that there are two intersection points between $c_b(\alpha)$ and $c^\dagger(\alpha)$, i.e., $(0,0)$ and (c_x,α_x) (where $c_x \approx 0.0231$ and $\alpha_x \approx 0.0117$). Thus, $c_b(\alpha)$ is properly defined and increasing on $(0,\alpha_x)$.

It can be shown that $c_a(\alpha)$ is also passing through (c_x, α_x), thus, $c_a(\alpha)$ is properly defined on $\left(\alpha_x, \frac{1}{17}\right)$.

· Case 1-i-b-II: $\frac{1}{17} \le \alpha < \frac{1}{16}$.

In this case, we have $\frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right) \le c^{\dagger}(\alpha)$. Thus, the profit difference between S and *CE-PL* is:

$$
\begin{aligned}
H_{S,2} \triangleq \Big(& \sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)}\left(4\alpha^2 + \left(17\alpha^2 - 18\alpha + 1\right)c^2 + 4\alpha(9\alpha - 1)c\right) \\
& - \left(-8\alpha^3 + \left(71\alpha^3 - 109\alpha^2 + 37\alpha + 1\right)c^3 + 2\alpha\left(109\alpha^2 - 74\alpha - 3\right)c^2 + 4\alpha^2(37\alpha + 3)c\right)\Big) \\
& / \left(64(1-\alpha)\alpha(2\alpha - c)(2\alpha + (\alpha - 1)c)\right) \\
& - \frac{2\alpha + \alpha^2(c+6) - 4\sqrt{\alpha^2(\alpha+1)(2\alpha + (\alpha-1)c)} - c}{(1-\alpha)^2}.
\end{aligned}
$$

Similarly, we can simplify $H_{S,2}(\alpha, c)$ and finally analyze $H_{S,3}(\alpha, c)$. Following the same step in case 1-i-b-I, we can get that in this region, $\frac{\partial H_{S,3}(\alpha,c)}{\partial c} < 0$. $\frac{\partial H_{S,3}(\alpha,c)}{\partial \alpha}$ changes signs. As such, it is not possible to characterize the threshold between S and *CE-PL* as a function of c (there exist values of c for which increasing α leads to multiple crossings between optimality regions for S and *CE-PL*).
Nevertheless, moving horizontally, given that $\frac{\partial H_{S,3}(\alpha,c)}{\partial c} < 0$, a threshold (crossing) boundary between optimality regions for *CE-PL* and S, within this particular region of the parameter space, is unique for every α, *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region.

We look at two particular cases for this region:
(1) First, we consider points on the boundary $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$. It can be shown that $H_{S,3}(\alpha, c)\Big|_{c=\frac{2\alpha(1-16\alpha)}{1-9\alpha}} > 0$. Thus, S dominates *CE-PL* on $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$.

(2) First, we consider points on the boundary $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$. It can be shown that $H_{S,3}(\alpha, c)\Big|_{c=\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)} < 0$. Thus, *CE-PL* dominates S on $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$.

Therefore, as we increase c, there can be at most one crossing point between optimality regions for S and *CE-PL*, which is defined as $c_a(\alpha)$ in case 1-i-b-I. Thus, we can further extend the domain of $c_a(\alpha)$ to $\left(\alpha_x, \frac{1}{16}\right)$.

– Case 1-ii: $\frac{1}{16} \le \alpha < \frac{1}{9}$.
In this case, $\frac{2\alpha(1-16\alpha)}{1-9\alpha} \le 0 \le c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. Therefore, $H_{S,1} \ge 0$. We square both sides of the inequality D.3 and follow the same step in case 1-i-b. The inequality D.3 is

equivalent to:

$$0 \le c < \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right).$$

Also, it can be shown that $\frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right) < c^{\dagger}(\alpha)$. Therefore, in this region, the profit difference between S and *CE-PL* is:

$$\begin{aligned} H_{S,2} \triangleq \Big(& \sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)}\left(4\alpha^2 + \left(17\alpha^2 - 18\alpha + 1\right)c^2 + 4\alpha(9\alpha - 1)c\right) \\ & - \left(-8\alpha^3 + \left(71\alpha^3 - 109\alpha^2 + 37\alpha + 1\right)c^3 + 2\alpha\left(109\alpha^2 - 74\alpha - 3\right)c^2 + 4\alpha^2(37\alpha + 3)c\right)\Big) \\ & / \left(64(1-\alpha)\alpha(2\alpha - c)(2\alpha + (\alpha - 1)c)\right) \\ & - \frac{2\alpha + \alpha^2(c+6) - 4\sqrt{\alpha^2(\alpha+1)(2\alpha + (\alpha-1)c)} - c}{(1-\alpha)^2}. \end{aligned}$$

Similarly, we can simplify $H_{S,2}(\alpha, c)$ and finally analyze $H_{S,3}(\alpha, c)$. Following the same step in case 1-i-b-I, we can get that in this region, $\frac{\partial H_{S,3}(\alpha,c)}{\partial \alpha} < 0$. $\frac{\partial H_{S,3}(\alpha,c)}{\partial c}$ changes signs. As such, it is not possible to characterize the threshold between S and *CE-PL* as a function of α (there exist values of α for which increasing c leads to multiple crossings between optimality regions for S and *CE-PL*).
Nevertheless, moving vertically, given that $\frac{\partial H_{S,3}(\alpha,c)}{\partial \alpha} < 0$, a threshold (crossing) boundary between optimality regions for *CE-PL* and S, within this particular region of the parameter space, is unique for every c, *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region.

We look at two particular cases for this region:
(1) First, we consider points on the boundary $\alpha = \frac{1}{16}$. It can be shown that $H_{S,3}(\alpha, c)\Big|_{\alpha=\frac{1}{16}} > 0$. Thus, S dominates *CE-PL* on $\alpha = \frac{1}{16}$.

(2) Then, we consider points on the boundary $\alpha = \frac{1}{9}$. It can be shown that $H_{S,3}(\alpha, c)\Big|_{\alpha=\frac{1}{9}} < 0$. Thus, *CE-PL* dominates S on $\alpha = \frac{1}{9}$.

Therefore, as we increase α, there can be at most one crossing point between optimality regions for S and *CE-PL*, then there exists a unique boundary, which we define as $\hat{\alpha}^{\dagger}(c)$, which separates the optimality regions for S and *CE-PL*. It satisfies:

$$H_{S,3}(\hat{\alpha}^{\dagger}(c), c) = 0.$$

It is straightforward that the domain of $\hat{\alpha}^{\dagger}(c)$ is $\left(0, c_a(\frac{1}{16})\right)$.

- Case 2: $\frac{1}{9} \le \alpha < \frac{1}{4}$.
  In this case, $H_{S,1}(\alpha, c)$ is decreasing in c, $32\alpha^2 + c - \alpha(9c+2) > 0$ is equivalent to $c < \frac{2\alpha(1-16\alpha)}{1-9\alpha}$. It can be shown that $\frac{2\alpha(1-16\alpha)}{1-9\alpha} > \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. Thus, $32\alpha^2 + c - \alpha(9c+2) > 0$. We square both sides of the inequality (D.3) and follow the same step in case 1-i-b. The inequality (D.3) is

equivalent to:

$$0 \le c < \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right).$$

Comparing $\frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$ with $\frac{2\alpha(1-4\alpha)}{1-3\alpha}$, we further get two cases:

- Case 2-i: $\frac{1}{9} \le \alpha < \frac{1}{17}\left(5 - 2\sqrt{2}\right)$.
  In this case, $\frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right) < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. It can be shown that $\frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right) < c^{\dagger}(\alpha)$. Therefore, in this region, the profit difference between S and *CE-PL* is:

$$\begin{aligned} H_{S,2} \triangleq \Big(&\sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)}\left(4\alpha^2 + \left(17\alpha^2 - 18\alpha + 1\right)c^2 + 4\alpha(9\alpha - 1)c\right) \\ &- \left(-8\alpha^3 + \left(71\alpha^3 - 109\alpha^2 + 37\alpha + 1\right)c^3 + 2\alpha\left(109\alpha^2 - 74\alpha - 3\right)c^2 + 4\alpha^2(37\alpha + 3)c\right)\Big) \\ &/\left(64(1-\alpha)\alpha(2\alpha - c)(2\alpha + (\alpha - 1)c)\right) \\ &- \frac{2\alpha + \alpha^2(c+6) - 4\sqrt{\alpha^2(\alpha+1)(2\alpha + (\alpha-1)c)} - c}{(1-\alpha)^2}. \end{aligned}$$

  It can be shown that $H_{S,3} < 0$, i.e., $H_{S,2} < 0$. Thus, S is dominated by *CE-PL*.

- Case 2-ii: $\frac{1}{17}\left(5 - 2\sqrt{2}\right) \le \alpha < \frac{1}{4}$.
  In this case, $\frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right) \ge \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. Comparing $\frac{2\alpha(1-4\alpha)}{1-3\alpha}$ with $c^{\dagger}(\alpha)$, we further have two cases:
  - Case 2-ii-a: $\frac{2\alpha(1-4\alpha)}{1-3\alpha} < c^{\dagger}(\alpha)$.
    The profit difference between S and *CE-PL* is:

$$\begin{aligned} H_{S,2} \triangleq \Big(&\sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)}\left(4\alpha^2 + \left(17\alpha^2 - 18\alpha + 1\right)c^2 + 4\alpha(9\alpha - 1)c\right) \\ &- \left(-8\alpha^3 + \left(71\alpha^3 - 109\alpha^2 + 37\alpha + 1\right)c^3 + 2\alpha\left(109\alpha^2 - 74\alpha - 3\right)c^2 + 4\alpha^2(37\alpha + 3)c\right)\Big) \\ &/\left(64(1-\alpha)\alpha(2\alpha - c)(2\alpha + (\alpha - 1)c)\right) \\ &- \frac{2\alpha + \alpha^2(c+6) - 4\sqrt{\alpha^2(\alpha+1)(2\alpha + (\alpha-1)c)} - c}{(1-\alpha)^2}. \end{aligned}$$

    It can be shown that $H_{S,2} < 0$. Thus, S is dominated by *CE-PL*.
  - Case 2-ii-b: $\frac{2\alpha(1-4\alpha)}{1-3\alpha} \ge c^{\dagger}(\alpha)$.
    The profit difference between S and *CE-PL* is:

$$\begin{aligned} H_{S,4} \triangleq \Big(&\sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)}\left(4\alpha^2 + \left(17\alpha^2 - 18\alpha + 1\right)c^2 + 4\alpha(9\alpha - 1)c\right) \\ &- \left(-8\alpha^3 + \left(71\alpha^3 - 109\alpha^2 + 37\alpha + 1\right)c^3 + 2\alpha\left(109\alpha^2 - 74\alpha - 3\right)c^2 + 4\alpha^2(37\alpha + 3)c\right)\Big) \\ &/\left(64(1-\alpha)\alpha(2\alpha - c)(2\alpha + (\alpha - 1)c)\right) \\ &- \frac{(c - 2\alpha)^2}{8\alpha}. \end{aligned}$$

    It can be shown that $H_{S,4} < 0$. Thus, S is dominated by *CE-PL*.

To summarize, we define $c^{\ddagger}(\alpha)$ as:

$$c^{\ddagger}(\alpha) \triangleq \begin{cases} c_a(\alpha) & \text{, if } \alpha_x \le \alpha < \frac{1}{16}, \\ c_b(\alpha) & \text{, if } 0 < \alpha < \alpha_x. \end{cases}$$

Then S dominates *CE-PL* if and only if:

$$\begin{array}{ll} 0 \le c < c^{\ddagger}(\alpha) & , \text{if } 0 < \alpha < \frac{1}{16}, \\ \quad \text{and} & \\ \frac{1}{16} < \alpha < \hat{\alpha}^{\dagger}(c). & \end{array}$$

□

Proof of Proposition 3

When $\alpha \ge 1$, by directly comparing profits and social welfare values from Propositions D.1-D.4, it can be easily seen that *CE-PL* is always the dominant strategy for the firm, whereas *TLF* is always the strategy that yields the highest social welfare.

The bulk of the proof, below, is addressing the considerably more complex case $\underline{0 < \alpha < 1}$.

Let us define:

$$\hat{\alpha}_1(c) \triangleq \quad \hat{\alpha}_f(c) \quad , \text{if } 0 \le c < c_4$$

$$\hat{\alpha}_2(c) \triangleq \begin{cases} \hat{\alpha}_e(c) & , \text{if } 0 \le c < c_4, \\ \hat{\alpha}_a(c) & , \text{if } c_4 \le c < c_1, \\ \hat{\alpha}_b(c) & , \text{if } c_1 \le c < c_2, \end{cases}$$

and

$$\hat{\alpha}_3(c) \triangleq \begin{cases} \hat{\alpha}_g(c) & , \text{if } 0 \le c < c_5, \\ \hat{\alpha}_d(c) & , \text{if } c_5 \le c < c_3, \\ \hat{\alpha}_c(c) & , \text{if } c_3 \le c < c_2, \end{cases}$$

and

$$\hat{\alpha}_4(c) \triangleq \begin{cases} \hat{\alpha}_g(c) & , \text{if } 0 \le c < c_5, \\ \hat{\alpha}^{\dagger}(c) & , \text{if } c_5 \le c < c^{\ddagger}(\frac{1}{16}), \end{cases}$$

where functions $\hat{\alpha}_a(\cdot)$, $\hat{\alpha}_b(\cdot)$, $\hat{\alpha}_c(\cdot)$, $\hat{\alpha}_d(\cdot)$, $\hat{\alpha}_e(\cdot)$, $\hat{\alpha}_f(\cdot)$, $\hat{\alpha}_g(\cdot)$, as well as constant thresholds c_1, c_2, c_3, c_4, and c_5 are defined and further analyzed below. $\hat{\alpha}^{\dagger}(c)$ and $c^{\ddagger}(\alpha)$ are defined in the Prop D.4. For ease of identification, Figure D.1 contains the illustration of these boundaries and thresholds (this is a more detailed version of Figure 3 from the main body).

- **Definition of c_1 and $\hat{\alpha}_a(c)$. Monotonicity of $\hat{\alpha}_a(c)$.**
  We first compare *CE-PL* and *TLF* under the intersection of regions $0 \le c < \frac{\alpha}{2}$ and $13-4\sqrt{10} \le \alpha < 1$, it can immediately follws that this is a non-empty region. In this region, define the difference between optimal profits under *CE-PL* and *TLF* as:

$$\Psi_{a,D}(\alpha, c) \triangleq \frac{(c-2\alpha)^2}{8\alpha} - \frac{1}{4}.$$

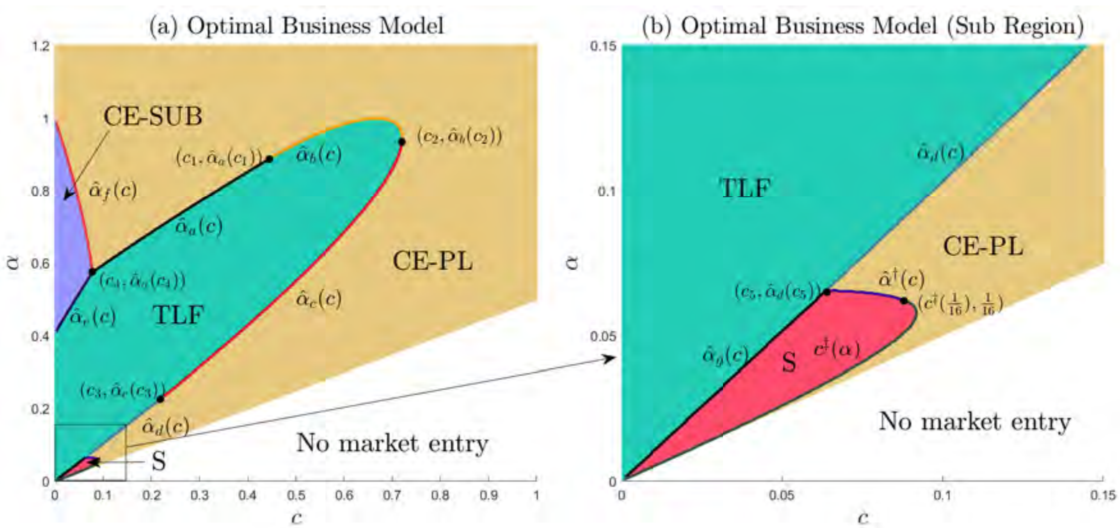

Figure D.1: Adoption Costs Scenario - Optimal Business Model - Marked Boundaries

We can obtain that:

$$\frac{\partial \Psi_{a,D}(\alpha, c)}{\partial \alpha} = \frac{1}{2} - \frac{c^2}{8\alpha^2} > 0,$$
$$\frac{\partial \Psi_{a,D}(\alpha, c)}{\partial c} = \frac{1}{4}\left(\frac{c}{\alpha} - 2\right) < 0.$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-PL* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists.*

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = 13-4\sqrt{10}$ and $\alpha = 1$ and examine the sign of $\Psi_{a,D}(\alpha, c)$ along these boundaries.

– On the boundary $\alpha = 13 - 4\sqrt{10}$, we obtain:

$$\Psi_{a,D}(\alpha, c)\Big|_{\alpha=13-4\sqrt{10}} = \frac{1}{72}\left(4\sqrt{10}+13\right)\left(c+8\sqrt{10}-26\right)^2 - \frac{1}{4} < 0.$$

– On the boundary $\alpha = 1$, we obtain:

$$\Psi_{a,D}(\alpha, c)\Big|_{\alpha=1} = \frac{1}{8}((c-4)c+2) > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\hat{\alpha}_a(c)}$, which separates the optimality regions for *CE-PL* and *TLF*. It satisfies:

$$\frac{(c-2\hat{\alpha}_a(c))^2}{8\hat{\alpha}_a(c)} - \frac{1}{4} = 0.$$

Also, it is straightforward that $\frac{\partial \hat{\alpha}_a(c)}{\partial c} = -\frac{\frac{\partial \Psi_{a,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{a,D}(\alpha,c)}{\partial \alpha}} > 0$. Hence, $\hat{\alpha}_a(c)$ is increasing in c.

It is straightforward there is a unique intersection point between $\hat{\alpha}_a(c)$ and $c = 0$, i.e., $(0, \frac{1}{2})$. Moreover, there is a unique intersection point between $\hat{\alpha}_a(c)$ and $\alpha = 2c$, i.e., $(c_1 = \frac{4}{9}, \frac{8}{9})$. Thus, $\hat{\alpha}_a(c)$ is properly defined and increasing on $[0, c_1)$.

- **Definition of c_2, c_3, $\hat{\alpha}_b(c)$ and $\hat{\alpha}_c(c)$. Monotonicity of $\hat{\alpha}_c(c)$.**
We then compare *CE-PL* and *TLF* under the intersection of regions $\frac{\alpha}{2} \le c < \alpha$ and the union of regions $0 < \alpha < 13 - 4\sqrt{10}, c^\dagger \le c < \alpha$ and $13 - 4\sqrt{10} \le \alpha < 1$ (In this union of regions, $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$). In this region, define the difference between optimal profits under *CE-PL* and *TLF* as:

$$\Psi_{b,D}(\alpha, c) \triangleq \frac{(c - 2\alpha)^2}{8\alpha} - \frac{c\left(1 - \frac{c}{\alpha}\right)}{\alpha}.$$

We can obtain that:

$$\frac{\partial \Psi_{b,D}(\alpha, c)}{\partial \alpha} = \frac{1}{2} - \frac{c((\alpha + 16)c - 8\alpha)}{8\alpha^3},$$
$$\frac{\partial \Psi_{b,D}(\alpha, c)}{\partial c} = \frac{(\alpha + 8)c - 2\alpha(\alpha + 2)}{4\alpha^2}.$$

As it turns out, in this range of the parameter space, $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial \alpha}$ and $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial c}$ changes signs. As such, it is not possible to characterize the threshold between *CE-PL* and *TLF* as a function of c or α.

Nevertheless, we first find the point that satisfies the equation $\Psi_{b,D}(\alpha, c) = 0$ (i.e., on the boundary between *CE-PL* and *TLF*) and has a vertical tangent line, i.e., $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial \alpha} = 0$. We can obtain that with in the above mentioned region, there is only one point that satisfies the condition, which is $\left(20\sqrt{5} - 44, 4\left(\sqrt{5} - 2\right)\right)$. We define $c_2 = 20\sqrt{5} - 44, 4\left(\sqrt{5} - 2\right)$.

Next, we define $\hat{\alpha}_b(c)$ and $\hat{\alpha}_c(c)$ by splitting the boundary $\Psi_{b,D}(\alpha, c) = 0$ at $\left(20\sqrt{5} - 44, 4\left(\sqrt{5} - 2\right)\right)$.

It is straightforward that when $c > 20\sqrt{5} - 44$, $\Psi_{b,D}(\alpha, c) > 0$, i.e., *CE-PL* dominates *TLF*. Then, we focus on the case when $c \le 20\sqrt{5} - 44$.

We first construct a line go through $\left(0, \frac{1}{2}\right)$ and $\left(20\sqrt{5} - 44, 4\left(\sqrt{5} - 2\right)\right)$. And it is straightforward that the expression of the line is: $\alpha_{l1} = \frac{\left(8\sqrt{5}-17\right)c}{8\left(5\sqrt{5}-11\right)} + \frac{1}{2}$.

If $\alpha_{l1} \leq \alpha < 2c$, we obtain that:

$$\begin{aligned} \frac{\partial \Psi_{b,D}(\alpha, c)}{\partial \alpha} &= \frac{1}{2} - \frac{c((\alpha+16)c - 8\alpha)}{8\alpha^3} > 0, \\ \frac{\partial \Psi_{b,D}(\alpha, c)}{\partial c} &= \frac{(\alpha+8)c - 2\alpha(\alpha+2)}{4\alpha^2}. \end{aligned}$$

As it turns out, in this range of the parameter space, $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial c}$ changes signs. Nevertheless, moving vertically, given that $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial \alpha} > 0$, a threshold (crossing) boundary between optimality regions for *CE-PL* and *TLF*, within this particular region of the parameter space, is unique for every c, if *it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = \alpha_{l2} = \frac{1}{4}\left(\sqrt{5}+3\right)c$ and $\alpha = 1$, and examine the sign of $\Psi_{b,D}(\alpha, c)$ along these boundaries.

- On the boundary $\alpha = \alpha_{l2}$, we obtain:

$$\Psi_{b,D}(\alpha, c)\Big|_{\alpha=\alpha_{l2}} = \frac{c\left(c\left(\left(9117 - 4077\sqrt{5}\right)c - 358912\sqrt{5} + 802608\right) + 64\left(3881\sqrt{5} - 8679\right)\right) + 512\left(123 - 55\sqrt{5}\right)}{64\left(\left(8\sqrt{5} - 17\right)c + 20\sqrt{5} - 44\right)^2} < 0.$$

- On the boundary $\alpha = 1$, we obtain:

$$\Psi_{b,D}(\alpha, c)\Big|_{\alpha=1} = \frac{1}{8}(2-3c)^2 > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\hat{\alpha}_b(c)}$, which separates the optimality regions for *CE-PL* and *TLF*. It satisfies:

$$\frac{(c - 2\hat{\alpha}_b(c))^2}{8\hat{\alpha}_b(c)} - \frac{c\left(1 - \frac{c}{\hat{\alpha}_b(c)}\right)}{\hat{\alpha}_b(c)} = 0.$$

It is easy to obtain that $(c_1, \frac{8}{9})$ is on $\hat{\alpha}_b(c)$. Thus, $(c_1, \frac{8}{9})$ is the unique interaction point between $\alpha = 2c$ and $\hat{\alpha}_b(c)$. Thus, $\hat{\alpha}_b(c)$ is properly defined on $[c_1, c_2)$.

Then we construct a line go through $(0,0)$ and $\left(20\sqrt{5} - 44, 4\left(\sqrt{5} - 2\right)\right)$. And it is straightforward that the expression of the line is: $\alpha_{l2} = \frac{1}{4}\left(\sqrt{5}+3\right)c$.

We can obtain that when $\alpha_{l2} \leq \alpha < \alpha_{l1}$, $\Psi_{b,D}(\alpha, c) < 0$, *TLF* dominates *CE-PL*.

If $c \leq \alpha < \alpha_{l2}$, we obtain that:

$$\begin{aligned} \frac{\partial \Psi_{b,D}(\alpha, c)}{\partial \alpha} &= \frac{1}{2} - \frac{c((\alpha+16)c - 8\alpha)}{8\alpha^3} < 0, \\ \frac{\partial \Psi_{b,D}(\alpha, c)}{\partial c} &= \frac{(\alpha+8)c - 2\alpha(\alpha+2)}{4\alpha^2} > 0. \end{aligned}$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-PL* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = c$ and $\alpha = \alpha_{l2} = \frac{1}{4}\left(\sqrt{5}+3\right)c$ and examine the sign of $\Psi_{b,D}(\alpha, c)$ along these boundaries.

– On the boundary $\alpha = c$, we obtain:

$$\Psi_{b,D}(\alpha, c)\Big|_{\alpha=c} = \frac{c}{8} > 0.$$

– On the boundary $\alpha = \alpha_{l2}$, we obtain:

$$\Psi_{b,D}(\alpha, c)\Big|_{\alpha=\alpha_{l2}} = \frac{c}{4} - 5\sqrt{5} + 11 > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\hat{\alpha}_c(c)}$, which separates the optimality regions for *CE-PL* and *TLF*. It satisfies:

$$\frac{(c-2\hat{\alpha}_c(c))^2}{8\hat{\alpha}_c(c)} - \frac{c\left(1-\frac{c}{\hat{\alpha}_c(c)}\right)}{\hat{\alpha}_c(c)} = 0.$$

Also, it is straightforward that $\frac{\partial \hat{\alpha}_c(c)}{\partial c} = -\frac{\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial \alpha}} > 0$. Hence, $\hat{\alpha}_c(c)$ is increasing in c.

Moreover, by solving the system of equations $c^\dagger = 0$ and $\frac{(c-2\hat{\alpha}_c(c))^2}{8\hat{\alpha}_c(c)} - \frac{c\left(1-\frac{c}{\hat{\alpha}_c(c)}\right)}{\hat{\alpha}_c(c)} = 0$, we can get there is a unique intersection point (it is around $(0.2255, 0.2329)$) between $c^\dagger$ and $\hat{\alpha}_c(c)$, denote it as $(c_3, \hat{\alpha}_c(c_3))$. Thus, $\hat{\alpha}_c(c)$ is properly defined and increasing on $[c_3, c_2)$.

- **Definition and Monotonicity of $\hat{\alpha}_d(c)$.**
  We then compare *CE-PL* and *TLF* under the intersection of regions $\frac{\alpha}{2} \le c < \alpha$ and $0 < \alpha < 13 - 4\sqrt{10}$, $0 \le c < c^\dagger$, it can immediately follows that this is a non-empty region. In this region, define the difference between optimal profits under *CE-PL* and *TLF* as:

$$\Psi_{d,D}(\alpha, c) \triangleq \frac{2\alpha + \alpha^2(c+6) - 4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)} - c}{(\alpha-1)^2} - \frac{c\left(1-\frac{c}{\alpha}\right)}{\alpha}.$$

First, we can obtain that in this region, when $\frac{\alpha}{2} \le c < \frac{2\alpha}{3}$, $\Psi_{d,D}(\alpha, c) < 0$, i.e., *TLF* dominates *CE-PL*.

Next, we check the case when $\frac{2\alpha}{3} \le c < \alpha$. We can further obtain that:

$$\begin{aligned}\frac{\partial \Psi_{d,D}(\alpha,c)}{\partial \alpha} &= -\frac{c^2}{\alpha^3} + \frac{c(\alpha-c)}{\alpha^3} + \frac{-4\alpha - 2\alpha^2(c+6) + 8\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)} + 2c}{(\alpha-1)^3} \\ &\quad + \frac{2\alpha(c+6) - \frac{4\alpha(\alpha(c+2)+1)}{\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}} - 4\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)} + 2}{(\alpha-1)^2} < 0, \\ \frac{\partial \Psi_{d,D}(\alpha,c)}{\partial c} &= \frac{c}{\alpha^2} + \frac{\alpha^2 - \frac{2(\alpha^2-1)\alpha}{\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}} - 1}{(\alpha-1)^2} + \frac{c-\alpha}{\alpha^2} > 0.\end{aligned}$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-PL* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $c = \frac{2}{3}\alpha$ and $c = \alpha$ and examine the sign of $\Psi_{d,D}(\alpha, c)$ along these boundaries.

- On the boundary $c = \frac{2}{3}\alpha$, we obtain:

$$\Psi_{d,D}(\alpha,c)\Big|_{c=\frac{2}{3}\alpha} = \frac{2\alpha\left(\alpha(3\alpha+26) - 6\sqrt{6}\sqrt{\alpha(\alpha+1)(\alpha+2)} + 8\right) - 2}{9(\alpha-1)^2} < 0.$$

- On the boundary $c = \alpha$, we obtain:

$$\Psi_{d,D}(\alpha,c)\Big|_{c=\alpha} = \frac{\alpha\left(\alpha(\alpha+6) - 4\sqrt{\alpha(\alpha+1)^2} + 1\right)}{(\alpha-1)^2} > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\hat{\alpha}_d(c)}$, which separates the optimality regions for *CE-PL* and *TLF*. It satisfies:

$$\frac{2\alpha + \alpha^2(c+6) - 4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)} - c}{(\alpha-1)^2} - \frac{c\left(1-\frac{c}{\alpha}\right)}{\alpha} = 0.$$

Also, it is straightforward that $\frac{\partial \hat{\alpha}_d(c)}{\partial c} = -\frac{\frac{\partial \Psi_{d,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{d,D}(\alpha,c)}{\partial \alpha}} > 0$. Hence, $\hat{\alpha}_d(c)$ is increasing in c.

- **Definition and Monotonicity of** $\hat{\alpha}_e(c)$.
  It is easy to obtain that when $c \le \alpha < 2c$, $\pi_{TLF} > \pi_{CE-SUB}$, i.e., *TLF* dominates *CE-SUB*.

  We then compare *CE-SUB* and *TLF* under the intersection of regions $\alpha \ge 2c$. In this region, define the difference between optimal profits under *CE-SUB* and *TLF* as:

$$\Psi_{e,D}(\alpha,c) \triangleq p_{a,D}\left(2 - \frac{c+p_{a,D}}{\alpha} - \frac{c+p_{a,D}}{1+c+p_{a,D}-\frac{c+p_{a,D}}{\alpha}}\right) - \frac{1}{4}.$$

Then, using the Envelope theorem (since $p_{a,D} \in \left(\frac{\alpha-c}{2}, \alpha - c\right)$ maximize $\pi_{CE,SUB}$), we have:

$$\frac{\partial \Psi_{e,D}(\alpha,c)}{\partial \alpha} = \frac{p_{a,D}(c+p_{a,D})\left(\frac{c+p_{a,D}}{\left(-\frac{c+p_{a,D}}{\alpha}+c+p_{a,D}+1\right)^2}+1\right)}{\alpha^2} > 0,$$

$$\frac{\partial \Psi_{e,D}(\alpha,c)}{\partial c} = p_{a,D}\left(-\frac{1}{\alpha} - \frac{(1-\alpha)\alpha(c+p_{a,D})}{(\alpha+(\alpha-1)c+(\alpha-1)p_{a,D})^2} - \frac{1}{-\frac{c+p_{a,D}}{\alpha}+c+p_{a,D}+1}\right) < 0.$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-SUB* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists.*

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = 2c$ and $\alpha = 2c + \frac{1}{2}$ and examine the sign of $\Psi_{e,D}(\alpha, c)$ along these boundaries.

- On the boundary $\alpha = 2c$, we obtain:

$$\Psi_{e,D}(\alpha,c)\Big|_{\alpha=2c} = p_{a,D}\left(-\frac{2c(c+p_{a,D})}{2c^2+2cp_{a,D}+c-p_{a,D}} - \frac{c+p_{a,D}}{2c}+2\right) - \frac{1}{4} < 0.$$

  The above inequality is satisfied for all $p \in \left(\frac{\alpha-c}{2}, \alpha - c\right)$.

- On the boundary $2c + \frac{1}{2}$, we obtain:

$$\begin{aligned}\Psi_{e,D}(\alpha,c)\Big|_{2c+\frac{1}{2}} &= p_{a,D}\left(-\frac{2(c+p_{a,D})}{4c+1} - \frac{c+p_{a,D}}{-\frac{2(c+p_{a,D})}{4c+1}+c+p_{a,D}+1}+2\right) - \frac{1}{4} \\ &\geq \frac{6c\left(2c\left(8c^2+2c+1\right)+1\right)+1}{8(4c+1)(2c(12c+7)+3)} (\text{Plug } p_{a,D} = \frac{1}{4}(2c+1)) \\ &> 0.\end{aligned}$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\hat{\alpha}_e(c)}$, which separates the optimality regions for *CE-SUB* and *TLF*. It satisfies:

$$p_{a,D}\left(2 - \frac{c+p_{a,D}}{\hat{\alpha}_e(c)} - \frac{c+p_{a,D}}{1+c+p_{a,D}-\frac{c+p_{a,D}}{\hat{\alpha}_e(c)}}\right) - \frac{1}{4} = 0.$$

Also, it is straightforward that $\frac{\partial \hat{\alpha}_e(c)}{\partial c} = -\frac{\frac{\partial \Psi_{e,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{e,D}(\alpha,c)}{\partial \alpha}} > 0$. Hence, $\hat{\alpha}_e(c)$ is increasing in c.

- **Definition of c_4 and $\hat{\alpha}_f(c)$. Monotonicity of $\hat{\alpha}_f(c)$.**
  We further compare *CE-SUB* with *CE-PL*. From the definition of $\hat{\alpha}_a(c)$, we know that $\hat{\alpha}_a(c)$ has a unique interaction point with y axis, i.e., $(0, \frac{1}{2})$. Given that $\hat{\alpha}_a(c)$ is increasing in c, *CE-PL* can only have the possibility to become the dominant strategy when $\frac{1}{2} < \alpha < 1$. Also, from the definition of $\hat{\alpha}_e(c)$, we know that $\hat{\alpha}_e(c)$ has a unique interaction point with $\alpha = 1$,

i.e., $(1-\frac{1}{\sqrt{2}}, 1)$. *CE-SUB* can only have the possibility to become the dominant strategy when $0 \le c < 1 - \frac{1}{\sqrt{2}}$. Thus, we only need to compare *CE-SUB* with *CE-PL* in the intersection of $\frac{1}{2} \le \alpha < 1$, $0 \le c < 1 - \frac{1}{\sqrt{2}}$, and $\alpha \ge 2c$.

In this region, define the difference between optimal profits under *CE-SUB* and *CE-PL* as:

$$\Psi_{e,D}(\alpha, c) \triangleq p_{a,D}\left(2 - \frac{c + p_{a,D}}{\alpha} - \frac{c + p_{a,D}}{1 + c + p_{a,D} - \frac{c+p_{a,D}}{\alpha}}\right) - \frac{(c-2\alpha)^2}{8\alpha}.$$

Then, using the Envelope theorem (since $p_{a,D} \in \left(\frac{\alpha-c}{2}, \alpha - c\right)$ maximize $\pi_{CE,SUB}$), we have:

$$\frac{\partial \Psi_{f,D}(\alpha, c)}{\partial \alpha} = \frac{c^2 + 8cp_{a,D} + 8p_{a,D}^2}{8\alpha^2} + \frac{p_{a,D}(c + p_{a,D})^2}{(\alpha + (\alpha-1)c + (\alpha-1)p_{a,D})^2} - \frac{1}{2},$$

$$\frac{\partial \Psi_{f,D}(\alpha, c)}{\partial c} = -\frac{c}{4\alpha} + p_{a,D}\left(-\frac{1}{\alpha} + \frac{(\alpha-1)\alpha(c + p_{a,D})}{(\alpha + (\alpha-1)c + (\alpha-1)p_{a,D})^2} - \frac{1}{-\frac{c+p_{a,D}}{\alpha} + c + p_{a,D} + 1}\right) + \frac{1}{2} < 0.$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-SUB* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, let's check the sign of $\frac{\partial \Psi_{f,D}(\alpha,c)}{\partial \alpha}$. Bring all the terms to a common denominator, we can write $\frac{\partial \Psi_{f,D}(\alpha,c)}{\partial \alpha} = \frac{q_{1,D}}{q_{2,D}}$, where:

$$\begin{aligned}
q_{1,D} \triangleq\ & 8p_{a,D}^4(1-\alpha)^2 + 8p_{a,D}^3\left(\alpha(3\alpha-2) + 3(\alpha-1)^2c\right) \\
& + p_{a,D}^2\left(-4\alpha^2\left(\alpha^2 - 2\alpha - 1\right) + 25(\alpha-1)^2c^2 + 16\alpha(3\alpha-2)c\right) \\
& + 2p_{a,D}\left(-4(\alpha-1)\alpha^3 + 5(\alpha-1)^2c^3 + \alpha(13\alpha-9)c^2 - 4(\alpha-2)\alpha^3c\right) + (\alpha + (\alpha-1)c)^2\left(c^2 - 4\alpha^2\right), \\
q_{2,D} \triangleq\ & 8\alpha^2(\alpha + \alpha c - c + \alpha p_{a,D} - p_{a,D})^2 > 0.
\end{aligned}$$

Thus, the sign of $\frac{\partial \Psi_{f,D}(\alpha,c)}{\partial \alpha}$ is the same as the sign of the numerator, $q_{1,D}$. We use $G_{SUB,D}(p_{a,D}) = 0$ to reduce the expression of $q_{1,D}$ from a quartic polynomial in $p_{a,D}$ to a quatratic one, as follows:

$$\begin{aligned}
q_{1,D} = & \frac{1}{(1-\alpha)^2} \times \left(p_{a,D}^2(1-\alpha)\left(2\alpha^2(\alpha((\alpha-3)\alpha+10)+2) + (\alpha-1)^3c^2 - 4\alpha^2(1-\alpha)c\right)\right. \\
& + p_{a,D}\left(-4\alpha^3(\alpha((\alpha-4)\alpha+5)+2) - 2(\alpha-1)^4c^3 - 2\alpha(5\alpha-1)(\alpha-1)^2c^2 - 4\alpha^2(\alpha((\alpha-4)\alpha+9)+2)(\alpha-1)c\right) \\
& + 4\alpha^4(\alpha+1) - (\alpha-1)^4c^4 - 2\alpha(3\alpha-1)(\alpha-1)^2c^3 - \alpha^2(\alpha(2(\alpha-4)\alpha+15)+3)(\alpha-1)c^2 \\
& \left. -2(\alpha-5)\alpha^4(\alpha-1)c - 8\alpha^3c\right).
\end{aligned}$$

Denote:

$$\begin{aligned}
A &= (1-\alpha)\left(2\alpha^2(\alpha((\alpha-3)\alpha+10)+2)+(\alpha-1)^3c^2-4\alpha^2(1-\alpha)c\right)\\
B &= -4\alpha^3(\alpha((\alpha-4)\alpha+5)+2)-2(\alpha-1)^4c^3-2\alpha(5\alpha-1)(\alpha-1)^2c^2\\
&\quad -4\alpha^2(\alpha((\alpha-4)\alpha+9)+2)(\alpha-1)c\\
C &= 4\alpha^4(\alpha+1)-(\alpha-1)^4c^4-2\alpha(3\alpha-1)(\alpha-1)^2c^3-\alpha^2(\alpha(2(\alpha-4)\alpha+15)+3)(\alpha-1)c^2\\
&\quad -2(\alpha-5)\alpha^4(\alpha-1)c-8\alpha^3c.
\end{aligned}$$

Then, $q_{1,D}=\frac{1}{(1-\alpha)^2}\times\left(Ap_{a,D}^2+Bp_{a,D}+c\right)$. Define the quadratic function $H_{SUB,PL,D}(p)\triangleq Ap^2+Bp+c$. In this range of the parameter space, it can be shown that $B^2-4AC>0$ and $A>0$. Hence, there are two real solutions to the equation $H_{SUB,PL,D}(p)=0$, namely:

$$p_{H1}=\frac{-B-\sqrt{B^2-4AC}}{2A}\quad\text{and}\quad p_{H2}=\frac{-B+\sqrt{B^2-4AC}}{2A}.$$

It can be shown that $\frac{\alpha-c}{2}<p_{H1}<\alpha-c<p_{H2}$. Recall that $p_{a,D}$ is the unique solution of $G_{SUB,D}(p)=0$. Moreover, from the proof of Prop. D.2, we know that $G_{SUB,D}(p)>0$ on $(\frac{\alpha-c}{2},p_{a,D})$ and $G_{SUB,D}(p)<0$ on $(p_{a,D},\alpha-c)$. It can be proved directly that $G_{SUB,D}(p_{H1})>0=G_{SUB,D}(p_{a,D})$. Hence, $\frac{\alpha-c}{2}<p_{H1}<p_{a,D}<\alpha-c<p_{H2}$. Furthermore, it can be shown that $A>0$, which indicates that $\bar{H}_{SUB,PL}(p)$ is convex. Therefore, $H_{SUB,PL,D}(p)<0$. Hence, in this region of the parameter space:

$$\frac{\partial\Psi_{f,D}(\alpha,c)}{\partial\alpha}<0.$$

So far, we proved that a threshold (crossing) boundary between optimality regions for *CE-SUB* and *CE-PL* within this particular region of the parameter space is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists.*

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $c=0$ and $c=1-\frac{1}{\sqrt{2}}$ and examine the sign of $\Psi_{f,D}(\alpha,c)$ along these boundaries.

- On the boundary $c=0$, it defaults to our basic model. And from Prop 1, we know that $\pi^*_{CE-SUB}>\pi^*_{CE-PL}$, i.e., $\Psi_{f,D}(\alpha,c)\Big|_{c=0}>0$.
- On the boundary $c=1-\frac{1}{\sqrt{2}}$, we obtain:

$$\Psi_{f,D}(\alpha,c)\Big|_{c=1-\frac{1}{\sqrt{2}}}=p_{a,D}\left(\frac{-p_{a,D}+\frac{1}{\sqrt{2}}-1}{\alpha}+\frac{1}{\frac{1}{\alpha}+\frac{1}{-p_{a,D}+\frac{1}{\sqrt{2}}-1}-1}+2\right)-\frac{\left(4\alpha+\sqrt{2}-2\right)^2}{32\alpha}<0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\hat{\alpha}_f(c)}$, which separates the optimality regions for *CE-SUB* and *CE-PL*. It satisfies:

$$p_{a,D}\left(2-\frac{c+p_{a,D}}{\hat{\alpha}_e(c)}-\frac{c+p_{a,D}}{1+c+p_{a,D}-\frac{c+p_{a,E}}{\hat{\alpha}_e(c)}}\right)-\frac{(c-2\alpha)^2}{8\alpha}=0.$$

Also, it is straightforward that $\frac{\partial\hat{\alpha}_f(c)}{\partial c}=-\frac{\frac{\partial\Psi_{f,E}(\alpha,c)}{\partial c}}{\frac{\partial\Psi_{f,E}(\alpha,c)}{\partial\alpha}}<0$. Hence, $\hat{\alpha}_f(c)$ is decreasing in c.

As $\hat{\alpha}_e(c)$ is increasing in c, there exists a unique intersection point between $\hat{\alpha}_e(c)$ and $\hat{\alpha}_f(c)$. Defining this point as $(c_4,\hat{\alpha}_e(c_4))$. At this point, we get $\pi^*_{CE-SUB}=\pi^*_{CE-PL}$ (from the definition of $\hat{\alpha}_f(c)$) and $\pi^*_{CE-SUB}=\pi^*_{TLF}$ (from the definition of $\hat{\alpha}_e(c)$). Thus, $\pi^*_{CE-PL}=\pi^*_{TLF}$. $(c_4,\hat{\alpha}_e(c_4))$ is also on $\hat{\alpha}_a(c)$. $\hat{\alpha}_f(c)$ is properly defined and decreasing on $[0,c_4)$.

- **Definition of c_5 and $\hat{\alpha}_g(c)$. Monotonicity of $\hat{\alpha}_g(c)$.**
  We further compare *TLF* with *S*. From Proposition D.4, we know *S* dominates *CE-PL* when:

$$\begin{gathered}0\le c<c^{\ddagger}(\alpha) \quad \text{, if } 0<\alpha<\tfrac{1}{16},\\ \text{and}\\ \tfrac{1}{16}\le\alpha<\hat{\alpha}^{\dagger}(c).\end{gathered}$$

  In this region, we further consider two regions:

  - Region 1: $0\le c<\frac{\alpha}{2}$.
    In this region, it can be shown that $\pi^*_S<\pi^*_{TLF}$, i.e., *TLF* dominates *S*.
  - Region 2: $\frac{\alpha}{2}\le c<\alpha$.
    In this region, define the difference between optimal profits under *S* and *TLF* as:

$$\begin{aligned}\Psi_{g,D}(\alpha,c)\triangleq\Big(&\sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}\left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)c^2+4\alpha(9\alpha-1)c\right)\\&-\left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3+2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big)\\&/\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right)-\frac{c\left(1-\frac{c}{\alpha}\right)}{\alpha}.\end{aligned}$$

    First, it can be shown that when $\frac{\alpha}{2}\le c<\frac{3\alpha}{4}$, $\Psi_{f,D}(\alpha,c)<0$, i.e., *TLF* dominates *S*.

    Then we focus on the region $\frac{3\alpha}{4}\le c<\alpha$. We obtain that:

$$\begin{aligned}\frac{\partial\Psi_{g,D}(\alpha,c)}{\partial\alpha}&<0,\\ \frac{\partial\Psi_{g,D}(\alpha,c)}{\partial c}&>0.\end{aligned}$$

    Therefore, a threshold (crossing) boundary between optimality regions for *S* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists.*

    Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely

$c = \frac{3}{4\alpha}$ and $c = \alpha$ and examine the sign of $\Psi_{g,D}(\alpha, c)$ along these boundaries.

* On the boundary $c = \frac{3}{4\alpha}$, we obtain:

$$\Psi_{g,D}\Big|_{c=\frac{3}{4\alpha}} = \frac{\alpha\left(639\alpha^2 - 51\sqrt{\alpha^2(3\alpha+5)(51\alpha+5)} + 330\alpha + 215\right) - 5\sqrt{\alpha^2(3\alpha+5)(51\alpha+5)}}{1280(\alpha-1)\alpha} < 0.$$

  Thus, on the boundary, *TLF* dominates *S*.

* On the boundary $c = \alpha$, it is easy to get $\pi^*_{TLF} \to 0$, whereas $\pi^* S > 0$. Thus, $\pi^*_S > \pi^*_{TLF}$, *S* dominates *TLF*.

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\hat{\alpha}_g(c)}$, which separates the optimality regions for *S* and *TLF*. It satisfies:

$$\Psi_{g,D}(\hat{\alpha}_g(c), c) = 0.$$

Also, it is straightforward that $\frac{\partial \hat{\alpha}_g(c)}{\partial c} = -\frac{\frac{\partial \Psi_{g,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{g,D}(\alpha,c)}{\partial \alpha}} > 0$. Hence, $\hat{\alpha}_g(c)$ is decreasing in c.

As $\hat{\alpha}_g(c) > c$ and $c_1(\frac{1}{16}) \approx 0.0876 > \frac{1}{16}$, there exists a unique intersection point between $\hat{\alpha}_g(c)$ and $\hat{\alpha}^\dagger(c)$. Let us define this point as $(c_5, \hat{\alpha}_g(c_5))$. Then, $\hat{\alpha}_g(c)$ is properly defined and increasing on $[0, c_5)$.

Thus, we completely characterized lines $\hat{\alpha}^\dagger(c)$, $c_a(\alpha)$, $\hat{\alpha}_1(c)$, $\hat{\alpha}_2(c)$, $\hat{\alpha}_3(c)$ and $\hat{\alpha}_4(c)$, (in particular, segments, $\hat{\alpha}_a(c)$, $\hat{\alpha}_b(c)$, $\hat{\alpha}_c(c)$, $\hat{\alpha}_d(c)$, $\hat{\alpha}_e(c)$, $\hat{\alpha}_f(c)$, $\hat{\alpha}_g(c)$, as well as constant thresholds c_1, c_2, c_3, c_4, c_5).

Comparison of $\hat{\alpha}_1(c)$ and $\hat{\alpha}_2(c)$:
When $0 \le c < c_4$, we already showed that $\hat{\alpha}_f(c)$ is decreasing in c and $\hat{\alpha}_e(c)$ is increasing in c. Furthermore, these two lines intersect at the point $(c_4, \hat{\alpha}_e(c)[\hat{\alpha}_f(c)])$. Thus, we have $\hat{\alpha}_f(c) \ge \hat{\alpha}_e(c)$, i.e., $\hat{\alpha}_1(c) > \hat{\alpha}_2(c)$.

Comparison of $\hat{\alpha}_2(c)$ and $\hat{\alpha}_3(c)$:
The support for $\hat{\alpha}_g(c)$ is defined within $0 < \alpha < \frac{1}{9} < \hat{\alpha}_e(c)$. Thus, $\hat{\alpha}_g(c) < \hat{\alpha}_e(c)$. For the other segments of $\hat{\alpha}_2(c)$ and $\hat{\alpha}_3(c)$, we have to compare *TLF* and *CE-PL*. Thus, it immediately follows that $\hat{\alpha}_2(c) > \hat{\alpha}_3(c)$.

Derivation of the dominating strategy in the entire region $0 < \alpha < 1$:

- By the definition of $\hat{\alpha}_f(c)$ and $\hat{\alpha}_e(c)$, we know that when $\hat{\alpha}_2(c) \le \alpha < \hat{\alpha}_1(c)$, *CE-SUB* dominates *CE-PL* and *TLF*. *S* can only dominates *CE-PL* within a subregion in $0 < \alpha < \frac{1}{9}$. Thus, *CE-SUB* also dominates *TLF* as well when $\hat{\alpha}_2(c) \le \alpha < \hat{\alpha}_1(c)$.

- By the definition of $\hat{\alpha}_2(c)$ and $\hat{\alpha}_3(c)$ (including the comparison between *TLF* with *CE-SUB*, *CE-PL*, and *S*), it is straightforward to see that *TLF* is the optimal strategy when $\hat{\alpha}_3(c) \le \alpha < \hat{\alpha}_2(c)$.

- Building on Prop. D.4, we can get that *S* is the optimal strategy when $\alpha < \hat{\alpha}_3(c)$ and $c < c^\ddagger(\alpha)$.

This completes the mapping of dominant strategy to the parameter space (we also discussed the case $\alpha \geq 1$ at the very beginning of the proof). □

E Details of Extension 3 - Imperfect Self-Learning with 3 Periods

In this E-companion section, we present the details for the third robustness check, involving an extension of the main model to 3 periods and accommodating for imperfect self-learning.

As noted in Section 3.1, the baseline model treats TLF as a free trial followed by either a perpetual or a per-period subscription license - with only one post-trial period, these variants are equivalent. However, over a longer horizon (of 3 or more periods), the type of license that comes *after* the free trial makes a difference in the way adoption unfolds. As such, we retain the TLF notation for free trials followed by a *perpetual license*, and introduce a fifth go-to-market strategy, *TLF-SUB*, which entails *per-period subscription licensing* after the end of the free trial. We point out that, as of September 2025, the most prominent mobile app marketplaces, Apple App Store and Google Play Store, natively support TLF only for subscription-based apps (with TLF for one-time purchase apps being possible only through developer-implemented workarounds). Thus, distinguishing TLF from *TLF-SUB* is of practical importance for mobile app developers evaluating go-to-market options for products with multi-period adoption horizons.

This framework of imperfect self-learning also allows us to implement a more complex social learning model as well. Adopters are the ones spreading WOM at the end of any period of use. However, as different adopters can purchase for the first time a license in different periods, in the context of imperfect learning, there may be scenarios in which different past adopters end up with different updated priors at the same moment in time. For example, in the context of 3 periods, some adopters may purchase in period 1 while others purchase in period 2. Thus, at the end of period 2, some existing adopters may have been using the software for two periods (engaging in two consecutive periods of self-learning), whereas others experienced the software for just one period. As such, in the context of imperfect learning, non-adopters have to internalize outside signals with potentially different values, from different adopter groups.

To characterize the adoption trajectories, we add clarifying subscripts A or N to the priors for periods 2 and 3 to show the adoption *before* that respective period. For example, $a_{2,N}$ ($a_{2,A}$) represents the perceived quality at the beginning of period 2 among those period-1 non-adopters (adopters), and $a_{3,NA}$ ($a_{3,AA}$) represents the perceived quality at the beginning of period 3 among those adopters who have first adopted in period 2 (adopted in period 1 and continued to stay as adopters in period 2). In this illustrative extension we keep things simple and assume that customers who exit the market after previously adopting or trying the product for free (under *TLF*, *CE-SUB* or *TLF-SUB*) do not continue to send any more WOM signals afterwards (beyond the last period of use), do not engage in any subsequent valuation learning, and, as such, do not return to the market. We also assume that adopters stop paying attention to outside signals once they start using the product, pivoting entirely to self-learning from that point onwards. We leave it for future research to explore the many other possible combinations of learning and entering/exiting the market for settings with 3 or more periods.[E-1]

[E-1] This is not relevant for the setting in the main model with only 2 periods since WOM updating takes place at the end of period 1 and any learning during period 2 does not impact revenue.

Let us formalize the learning mechanisms in this extension. In the beginning of period 2, period-1 adopters' perceived valuation, updated via one period of *imperfect* self-learning is

$$a_{2,A} = a_1 + \delta(a - a_1), \tag{E.1}$$

in which $\delta > 0$ captures the degree of adjustment from imperfect self-learning. Our main setup is a special case of this extension setup, with $\delta = 1$, such that the adopters update their priors accurately to the true per-period value a after a single period of usage. When $\delta \in (0, 1)$, the adopters are learning from their usage, update in the correct direction, but, during a single period of use, "travel" only part of the distance between old prior and true value. When $\delta > 1$, the self-learning process can cause "overshooting," the adopters may excessively adjust their perceived utility upon usage past the true value (shifting from underestimators to overestimators or vice-versa). Entering period 3, we have to consider two subgroups of existing adopters - "veteran" adopters (those who adopted the product in both periods 1 and 2, whether freely or through a paid license - $N_{2,AA,total}$) and "recent" adopters (those who stayed out of the market in period 1 but adopted in period 2 - $N_{2,NA,total}$). Each of these subgroups engaged in imperfect self-learning in period 2, but they started from different priors (at the beginning of period 2). Their updated priors entering period 3 (at the end of period 2, after self-learning but before disseminating WOM) differ as follows:

$$a_{3,AA} = a_{2,A} + \delta(a - a_{2,A}), \tag{E.2}$$

$$a_{3,NA} = a_{2,N} + \delta(a - a_{2,N}), \tag{E.3}$$

in which $a_{2,A} = a_1 + \delta(a - a_1)$ is defined above in (E.1), and $a_{2,N} = a_1 + N_{1,total}^{\frac{1}{w}}(a_{2,A} - a_1)$ represents period 1 non-adopters' perceived quality factor at the beginning of period 2 (updated via social learning at the end of period 1). We assume non-adopters in period 2 are influenced by the average outside signal and the total volume of such signals. Specifically, these period 2 non-adopters are drawn towards a weighted average $a_{3,avg} = \frac{N_{2,AA,total} \times a_{3,AA} + N_{2,NA,total} \times a_{3,NA}}{N_{2,total}}$ of the outside signals from the two period 2 adopter groups, after their self-learning, with $N_{2,total} = N_{2,AA,total} + N_{2,NA,total}$. Thus, non-adopters in period 2 that are still in the market, enter period 3 with updated priors:

$$\begin{aligned} a_{3,NN} &= a_{2,N} + N_{2,total}^{\frac{1}{w}} \times (a_{3,avg} - a_{2,N}) \\ &= a_{2,N} \times (1 - N_{2,total}^{\frac{1}{w}}) + a_{3,avg} \times N_{2,total}^{\frac{1}{w}} \\ &= a_{2,N} \times (1 - N_{2,total}^{\frac{1}{w}}) + a_{3,AA} \times \frac{N_{2,AA,total}}{N_{2,total}} \times N_{2,total}^{\frac{1}{w}} + a_{3,NA} \times \frac{N_{2,NA,total}}{N_{2,total}} \times N_{2,total}^{\frac{1}{w}}. \end{aligned} \tag{E.4}$$

Note that under perfect self-learning ($\delta = 1$), both recent and veteran adopters send exactly the same signal (the true value) via WOM. As such, under $\delta = 1$, the social learning in (E.4) is consistent with the social learning in the main model in equation (1), for *any* strength w of WOM effects. An alternative candidate parameterization of $a_{3,NN}$ in which *each* outside signal value (as opposed to the average) receives separate weights similar to those in equation (1), while tempting to consider, leads to inconsistency with perfect learning.[E-2]

[E-2] It may feel tempting to consider a alternative different parameterization $a_{3,NN} = a_{2,N} + N_{2,AA,total}^{\frac{1}{w}} \times (a_{3,AA} - a_{2,N}) + N_{2,NA,total}^{\frac{1}{w}} \times (a_{3,AA} - a_{2,N}) = a_{2,N} \times (1 - N_{2,AA,total}^{\frac{1}{w}} - N_{2,NA,total}^{\frac{1}{w}}) + a_{3,AA} \times N_{2,AA,total}^{\frac{1}{w}} + a_{3,NA} \times N_{2,NA,total}^{\frac{1}{w}}$ as a direct extension of the social learning in equation (1). Note that the weights for the old prior and the outside signals end up different from the ones in (E.4). However, when $\delta = 1$, we would end up with *identical* outside WOM signals ($a_{3,AA} = a_{3,NA} = a$) weighted *differently*, depending on the origin of the signal. Thus, such an alternative

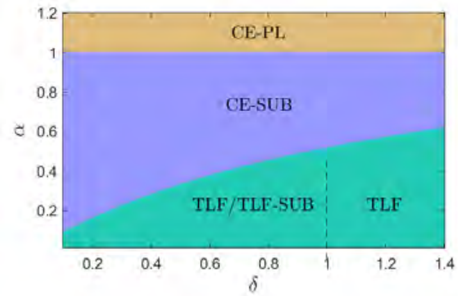

Figure E.1: Optimal strategies - model with 3 periods and imperfect self-learning ($w = 1$)

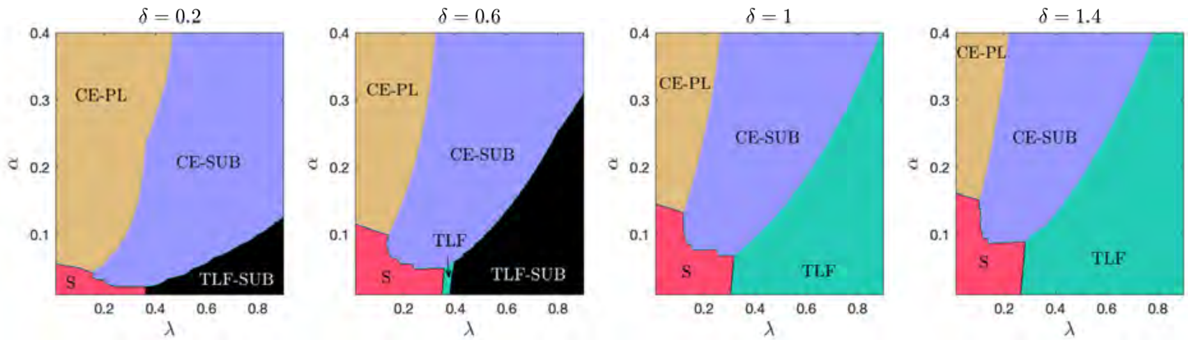

Figure E.2: 3-Period optimal strategies with imperfect self-learning and individual depreciation

In the numerical illustrations below, we consider intermediate strength of WOM effects ($w = 1$). However, qualitatively similar insights as in Section 6.2 hold under social learning of more general intensity (with stronger WOM effects inducing an increase the optimality region for S, in the presence of individual depreciation and/or adoption costs).

In the absence of individual depreciation and adoption costs, even under imperfect self-learning (regardless of the degree δ of such imperfect self-learning) and more than two periods, it turns out that S is still dominated in all regions of the parameter space. This can be seen in Figure E.1. We already covered previously the dynamics at play in the top part of the figure, when customers enter the market with intermediate to high priors. When initial priors are very low, free trials help ignite consumer self-learning. However, for obvious reasons, the benefit of free trials gets significantly diminished when the rate of self-learning, δ, is very low (bottom left corner) as customers do not update their WTP fast enough for the firm to be able to charge a higher price. In the absence of depreciation or adoption costs, when $\delta < 1$, TLF and $TLF\text{-}SUB$ yield identical outcomes. However, when $\delta > 1$, TLF strictly dominates $TLF\text{-}SUB$ as the firm will use the perpetual license for the remaining two periods to take advantage of consumers overshooting true value through self-learning and will charge a higher price, before they get another chance to recalibrate priors downwards.

Figures E.2 and E.3 illustrate our optimal go-to-market strategies in the current extension under individual depreciation and under adoption costs, respectively. This time, again, S emerges as the optimal strategy in bottom left corner. Our previous insights from Figures 2 and 3 remain qualitatively unchanged in the 3-period setting with imperfect self-learning. As long as $\delta < 1$, increasing the self-learning rate (towards perfect learning) will expand the optimality regions for S and, at the same time, will also lead to $TLF\text{-}SUB$ and TLF more or less trading places.

parameterization of $a_{3,NN}$ would be inconsistent with the main model with perfect self-learning.

Under individual depreciation, for higher λ, as more value remains in the second and third period, it matters how much of it has not yet been understood (through self-learning) after the free trial. With reduced self-learning (low δ as in panel (a)), there is still significant potential for self-learning to adjust priors substantially in period 2, after the free trial. As such, a per-period license after the free trial allows the firm to better monetize the tradeoff between continued self-learning and depreciation (*TLF-SUB* dominates *TLF*). However, if the self-learning rate is higher, there is less ground left to cover in period 2 in understanding the true value (as most of it has been learned during free trial). At the same time, even moderate depreciation starts eating more and more of the value in period 3 (as depreciation compounds). As such, at intermediate depreciation values, *TLF* starts overtaking *TLF-SUB*. As δ gets larger, consumers update their priors very close to the true value during the free trial, and depreciation will eventually dominate the benefits of continued self-learning in period 2. In period 3, we will start seeing churn of consumers under *TLF-SUB*. Regular *TLF*, with perpetual license, can better manage that scenario, as long as the depreciation is not too severe (in which case *S* will take over).

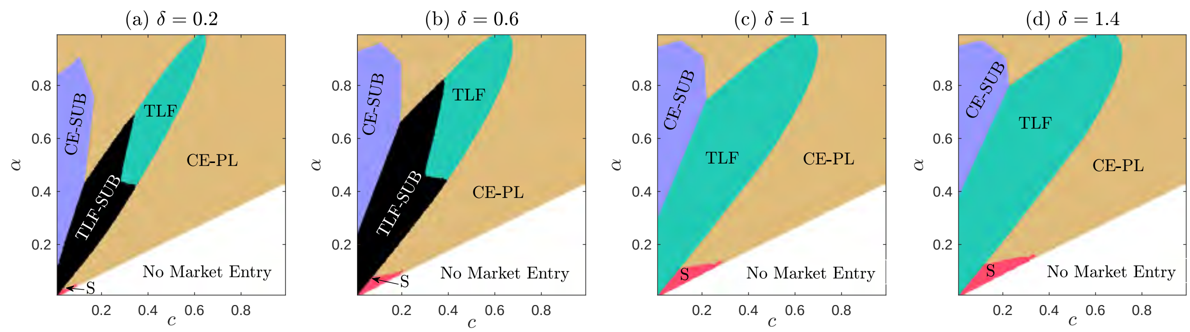

Figure E.3: 3-Period optimal strategies with imperfect self-learning and adoption costs

When accounting for adoption costs, we assume that the free trial is available only in period 1 - customers that skip it because of the adoption costs will no longer have the same opportunity in periods 2 or 3 to enroll in a free trial. Again *TLF-SUB* overtakes *TLF* for low self-learning rate and lower costs, but roles swap when self-learning rate gets higher and/or adoption costs get higher (but not too high). When costs are not too high *TLF-SUB* can better manage social learning for those customers that avoided the free trial due to adoption cost. However, when adoption costs are higher, it becomes gradually harder to woo anyone that skipped the free trial through WOM effects in a subsequent period. This is where offering a perpetual license instead of a per-period subscription license allows for better amortizing of these costs over time.

F Extension 4 - Model with Adoption Costs and *Endogenous* Individual Depreciation

We also explore a setting combining adoption costs with an endogenous depreciation rate, by making λ a decision variable. For the product to retain value beyond a single period of use, the firm will have to develop product features that continue to be relevant over time or commit to rejuvenating content. More precisely, in order for second-period continued use to generate a fraction λ of the original value for period 1 adopters, the firm will need to commit additional development cost $r(\lambda)$. We assume that $r(0) = 0$, i.e., with no additional effort, the product retains no value past a single period of use. We consider the development costs sunk for basic content or features that are associated with limited use and we assume that the firm is at a later stage in the development process when it has a minimum viable release candidate and it is deciding on whether to pack additional value into it. Moreover, consistent with the literature, we assume r is increasing convex (i.e., $r'(\lambda) > 0$ and $r''(\lambda) > 0$). In particular, for this numerical exploration, we consider a traditional quadratic cost function $r(\lambda) = \mu\lambda^2$, with $\mu > 0$. We present in Figure F.1 the optimal strategies under three different cost scenarios ($\mu \in \{0.001, 0.2, 0.5\}$). We confirm that all four strategies can be optimal. Resiliently, S can still emerge as the optimal choice under small α and small c when the rejuvenation cost spans low to high. When rejuvenation costs are low ($\mu = 0.001$), the outcome closely resembles the one depicted in Figure 3 as the firm has an incentive to choose high λ in all scenarios. Other dynamics previously discussed remain qualitatively robust (but the shapes of optimality regions change).

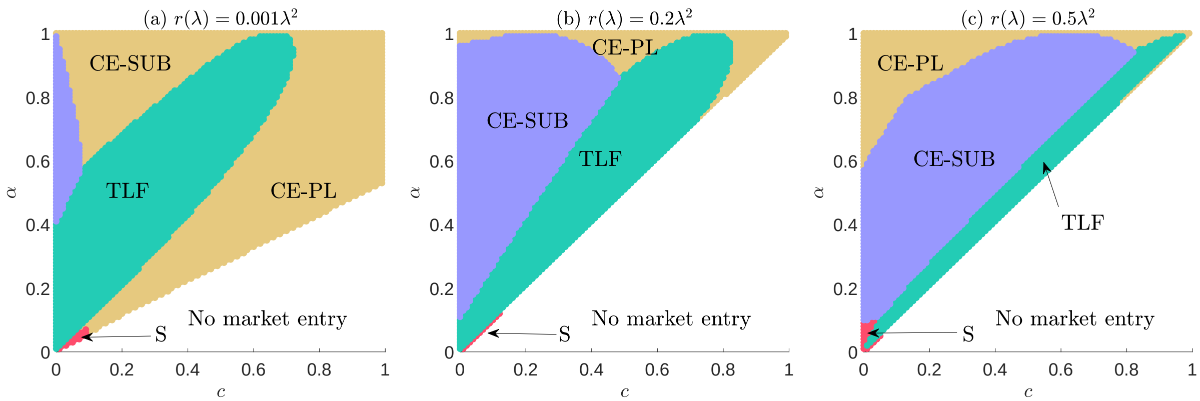

Figure F.1: Optimal strategies - model with adoption costs and endogenous individual depreciation

G Extension 5 - Model with Heterogeneous Consumer Priors

Previously, in the main body of the paper, we assumed that all consumers share the same prior factor α, implying that they either all underestimate or all overestimate their true product valuation when entering the market. In this section, we relax this assumption by allowing for heterogeneity in priors, whereby some customers initially underestimate the value of the product whereas others overestimate it. For $\alpha \in (0, 1)$, we consider a Bernoulli distribution for the consumer priors whereby a fraction τ of consumers (denoted by group H) initially overestimate the value of the product at a level $a_{H1} = (2 - \alpha)a$ and the other fraction $1 - \tau$ of consumers (denoted by group L) initially underestimate it at a level $a_{L1} = \alpha a$. The firm cannot tell group L and group H customers apart, but it has an understanding of τ. Intuitively, when $\tau = 0$ or $\tau = 1$, this setup reduces to the baseline model in Section 3.

Figure G.1 illustrates the optimal strategies when we extend the baseline model in Section 3 to account for heterogeneity in priors. In the absence of either individual depreciation or adoption costs, S is always dominated, consistent with prior results. This is due to the fact that the focus of S is mainly in generating WOM effects that enhance the valuation of consumers in group L. By uniformly seeding also some of the customers in group H, who initially overestimate the value of the

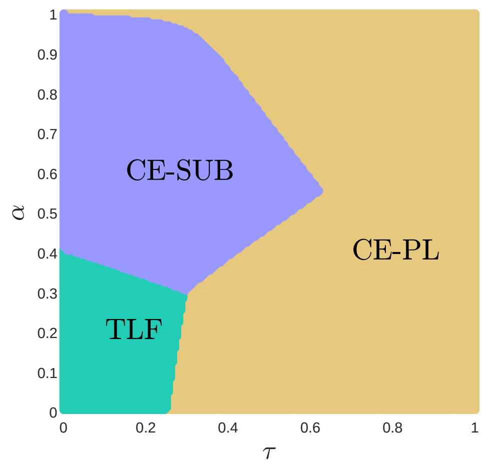

Figure G.1: Optimal strategies - baseline model with heterogeneous consumer priors

product and have inherently higher WTP, the firm would lose the ability to monetize these overestimating customers, along with seeded but higher-valuation underestimating customers. Moreover, WOM effects also impact negatively the perceived product valuation for the period 2 paying customers in group H that did not adopt in the first period (if any), further reducing their WTP in the second period. In contrast, *CE-PL* and *CE-SUB* can take better advantage of the existence of an overestimating subgroup of consumers by charging a premium price from the beginning before such customers have a chance to adjust downwards their priors through learning. *TLF* can only dominate when most consumers belong to group L, as it relies heavily on self-learning and monetization of second period, as we discussed before. We emphasize that this figure is different from prior figures as the y-axis reflects priors *only* for group L (α).

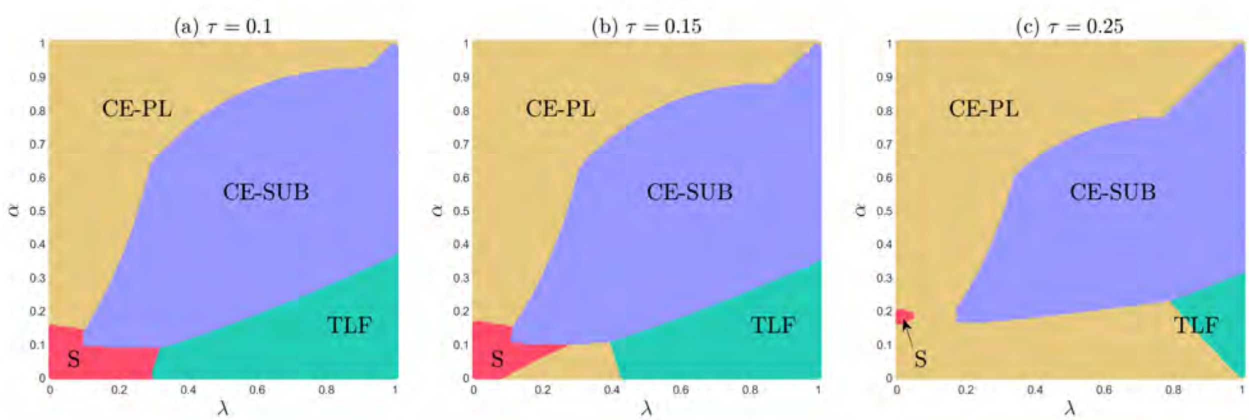

Figure G.2: Optimal strategy - model with individual depreciation and heterogeneous consumer priors

When we add individual depreciation to this setting (i.e., extend the model in Section 4 to include heterogeneous priors), we confirm the previous finding that all four models (including S) can emerge as optimal, as shown in Figure G.2. Nevertheless, the ability of S to dominate when customers underestimate the value of the product and there is significant depreciation hinges on most customers belonging to group L - when τ is small, in panel (a), the outcome resembles the one in Figure 2. As the size of group L shrinks and the size of group H increases, eventually seeding will no longer be optimal (we can see that the size of the optimality region for S is decreasing as τ increases - this region will eventually vanish). With homogeneous priors and depreciation (setup in Section 4), seeding was optimal for high depreciation (low λ) and heavy underestimation by *all* consumers. However, under heterogeneous priors, when α is very low, the valuation for group H is very large, diametrically opposite to that of group L relative to the true value. Hence, as the market share of group H grows, the firm will increasingly focus on pricing high for group H from the get-go (even to the extent of not serving group L at all) without sacrificing any of that high-valuation demand pool to generate WOM effects.

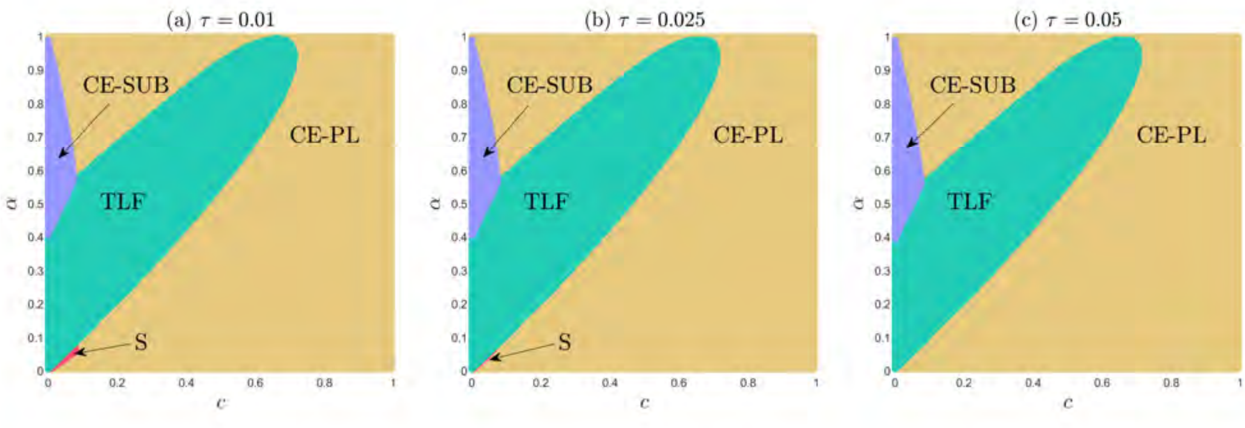

Figure G.3: Optimal strategies - model with adoption costs and heterogeneous consumer priors

Similarly, we can show that our results with adoption costs from Section 5 are qualitatively robust under the heterogeneous prior setting as well. In Figure G.3, we illustrate optimal strategies for $\tau \in \{0.01, 0.025, 0.05\}$. When group L is large (panels (a) and (b)), the outcome *almost* resembles the one in Figure 3, and S can emerge as the optimal strategy when α and c are low. Nevertheless, the region of optimality for S shrinks and vanishes relatively quickly as the size of group H increases, because that group is far less constrained by adoption costs and already has a high starting prior for the product, allowing the firm to price high without sacrificing revenue from seeded high valuation customers over two periods. However, different from the market outcome in Proposition 3, under the case when $\tau > 0$, the "no market entry" region shifts considerably to the right due to overestimating consumers. In particular, in the heterogeneous priors setup, the firm will stay out of the market iff the adoption cost is prohibitively high even for the overestimators (i.e., $c \geq 2(2-\alpha)$). Since the outcome is trivial for that region, it was cut off in the illustration, in order to focus on the more interesting regions.

H Related Literature on Free Trials (TLF)

The ability to influence consumer perceptions, purchase behavior, and dissemination of awareness through sampling campaigns has been recognized for a long time (Hamm et al. 1969, Holmes and Lett 1977, Goering 1985). Time-locked free trials and free demonstrations represent a special case of sampling where consumers get exposure to the full-feature product for a limited period of time. Heiman and Muller (1996) explore the optimal length of free trials and demonstrations in the context of physical goods, focusing in particular on cars and printers. In general, as physical goods and some digital services have marginal costs, it may not be optimal to cover the entire market through free trial campaigns. Accounting for such unit costs incurred by the vendor, Schlereth et al. (2013) and Li and Wang (2018) explore the optimal market coverage of free trial campaigns.

The literature on properties and performance of *TLF* go-to-market strategies in the digital space has progressed substantially in the last decade. Cheng and Liu (2012) and Dey et al. (2013) explore when it is optimal to offer *TLF* in software markets and how the length of the trial period should be calibrated. Cheng et al. (2015) compare *TLF* against other free sampling strategies (feature limited trials and hybrid feature/time limited trials). Wang and Özkan-Seely (2018) show that price can serve as a quality signal that complements direct experiential learning when *TLF* is the dominant strategy (the optimal trial length is positive). Datta et al. (2015) and Foubert and Gijsbrechts (2016) explore the impact of free trials on customer acquisition (conversion), churn, and overall customer lifetime value in the long run. Lee and Tan (2013) and Chen et al. (2017) investigate the interaction between WOM effects and free sampling strategies (including *TLF*) when exploring their market outcome. Sunada (2018) explores optimal free trial length in the presence of demand depreciation. Mehra and Saha (2018) study whether public betas and free trials should be used in tandem or not. Reza et al. (2021) explore how promotion redemption and subsequent usage are impacted when targeting existing users with hybrid time- and quantity-limited free trials. Wang et al. (2023) examine how the competitive use of *TLF* in a duopoly setting depends on both the magnitude of switching cost and the horizontal differentiation between firms. Yoganarasimhan et al. (2023) investigate personalized-length vs. uniform-length *TLF* strategies, and the impact of optimally-personalized free trials on short-term conversions and long-run customer loyalty and overall revenues.

While market-wide free trial strategies are not that common in the markets for physical goods, we do see widespread implementation of a seemingly similar, albeit quite different strategy - *free returns* (or full *money-back guarantees, MBGs* within a certain time frame) - occasionally paired with free shipping as well. Similar to free trials, *MBG* policies also help resolve consumer uncertainty and the risk of a mismatch, and may also positively impact consumer adoption decisions and willingness to pay a higher price (Che 1996, Suwelack et al. 2011, Bower and Maxham III 2012). Unlike with free trials and demonstrations, under *MBGs* consumers gain experience with the product after the purchase. At the same time, from the perspective of both consumers and providers/retailers, such strategies may add considerable costs. For consumers, there are inconvenience costs associated with the return process (repackaging the item, taking it to the retailer or a collection point, etc) and consumers must incur these costs in order to receive the refund (because, unlike in the case of free trials, consumers are charged upfront in the case of free return policies). Heiman et al. (2001) explore consumer preference for free demonstrations vs. *MBGs* and analyze scenarios when the two risk-reducing strategies complement or substitute each other. For providers/retailers, free returns add considerable logistical costs as well. Part of it is in terms of labor costs to process returns, which, alone, can in some cases cancel out the increase in revenue if we myopically consider short-term profits (Patel et al. 2021). In addition, goods used and returned during the free returns window in many cases exhibit wear and tear and cannot be re-commercialized as new items. The salvage value of returns represents an important factor in the implementation of *MBGs* for physical goods (Davis et al. 1995, Akçay et al. 2013) - some returns are unopened or in like-new condition and can be put back on the shelf right away, others can be refurbished/recertified and sold at a discount, and some necessitate retiring altogether, with the retailer (along with other entities upstream in the supply chain) absorbing the overall cost associated with the retired item. Moorthy and Srinivasan (1995) explore how offering *MBGs* can also be used to signal product quality. Furthermore, cross-channel full-refund or partial-refund returns (e.g., buy online, return in person) have been considered as a feature to influence consumer adoption in omnichannel operations and, potentially, help fight competition (He et al. 2020, Jin et al. 2020, Nageswaran et al. 2020).

In the context of digital goods and services, and more specifically software applications as well as online services, the provider costs associated with offering time-locked free trials become negligible. Once the product is built, inserting code to lock the product or service access upon the expiration of the free trial can be done with very few resources. As such, it is feasible to offer market wide free trials (*TLF*). Also, similarly, for this specific category of products and services, the costs of offering *MBGs* are negligible - once a user requests money back within an acceptable window after purchase, it is very easy for the provider to reverse the online transaction. There are no actual physical or digital returns for the products in this space - the license gets deactivated or the online access is revoked. In software application and services markets, both *TLF* and *MBG* strategies are employed.[H-1] One difference between *TLF* and *MBG* is that with *MBG* the customer pays upfront, whereas with *TLF*, in many cases consumers can download the free trial without initiating payment or providing details on how the payment will be processed (e.g., providing a credit card account). Arguably, with *TLF*, more consumers can try the product even if they cannot afford the paid version. However, with financial instruments that offer short-term access to capital (e.g., a credit card), even such customers can try products offered with *MBGs*. Another difference is that *TLF* is usually implemented by the developer and can benefit all consumers equally. On the other hand, *MBG* strategies are traditionally point-of-sale (retailer) strategies and can differ in extent across developer and various resellers of the same digital product.[H-2] Since in this paper we focus on a single decision-making vendor for the product, this difference is irrelevant for our analysis. As such, in the context of this study, in contrast to physical goods markets, in digital goods markets the aforementioned two risk-reducing strategies - *MBG* and *TLF* - are more or less equivalent. Given that, throughout this study we stick to *TLF* terminology.

I Motivation for Including and Distinguishing Between Perpetual and Subscription-Based Licenses; Relevance for the Mobile App Sector

In this E-companion section, we explain the rationale as to why, in the context of our framework and analysis, we differentiate between subscription and perpetual license models for paid models without free consumption (*CE-PL* and *CE-SUB*) and also for free trials (*TLF* vs *TLF-SUB* in Extension 3 in Section 6.3 and E-companion E). Moreover, we highlight how the inclusion of these strategies is particular relevant for the mobile app sector.

[H-1]Intuit Quickbooks and TaxAct desktop versions, as well as Autodesk online services all come with risk-free *MBGs*. We already discussed several *TLF* examples in the Introduction.

[H-2]For example, while offering a 15-30 day *MBG* for direct purchases, Autodesk (2025) also stated that, as of December 2025, “Return policies for subscription and subscription renewal charges from third-party retailers or authorized Autodesk partners vary.”

- Motivation for including both *CE-PL* and *CE-SUB* and distinguishing between them. Relevance for the mobile app sector.

  The strategies that involve *no free consumption* (*CE-PL* and *CE-SUB*) are included for completeness of the argument. They are both observed in practice and, overall, the second research objective (as defined on page 4 in the Introduction, as "*to identify specific factors that, when accounted for, support market scenarios under which S is the optimal strategy, even in the presence of TLF*") would be pretty moot if, in the regions in which S could dominate TLF (in the presence of the additional factors), S would in turn end up dominated by strategies with no free consumptions. What we really want to explore is when S is the optimal strategy, even with TLF included in the choice set *alongside* other traditional strategies without free consumption. It is this wider comparison that makes our results more relevant from a practical standpoint.

  CE-PL is the earlier, more established monetizing model for software (even before Internet was open to the public). *CE-SUB* is a more recent model, with many digital subscriptions being electronically renewed via the Internet. Some of the prior relevant studies focus only on one of the models (e.g., Niculescu and Wu 2014 focus only on *CE-PL*). In particular, the inclusion of *CE-SUB* in the analysis is of great importance to the exploration of go-to-market strategies for the mobile app sector. In June 2016, Apple announced that they were "opening auto-renewable *subscriptions* to all app categories including games, increasing developer revenue for eligible subscriptions after one year, providing greater pricing flexibility, and more." (Source: Apple - Developer News, `https://developer.apple.com/news/?id=06082016a`, accessed on 12/23/2025). Before that, Apple restricted this monetization mode solely to content apps such as magazines, newspapers, music/video streaming. For clarification, Google allowed in the Google Play Store these capabilities since 2012. Hence, since 2016, the two most prominent mobile app marketplaces became aligned in supporting widely the subscription model. Thus, since 2016, mobile app developers can utilize two distinct paid models with no free consumption - *CE-PL* and *CE-SUB* - *across all major mobile app ecosystems*. Therefore, it is now of high practical relevance to the mobile app sector (an important segment of the software market) to consider *both* of these options within the developers' choice set.

  From our analysis, it can be seen that in many cases, the optimality region of *CE-SUB* borders that of S or TLF. That means implicitly that *CE-SUB* is stronger that *CE-PL* in those regions and, as such, it directly erodes further into the optimality regions of S and/or TLF. Consequently, it is an even stronger insight that S can emerge optimal under individual depreciation or adoption costs when both paid strategies without free consumption are part of the developer's strategy choice set (relative to a setting without *CE-SUB*).

- Motivation for distinguishing between *TLF* (which is free trial followed by perpetual license) and *TLF-SUB* (which is free trial followed by subscription mode, as introduced in Extension 3 in Section 6.3 and E-companion E). Relevance for the mobile app sector.

In December 2017, Apple started allowing developers to implement TLF via "discounted introductory price" for an auto-renewable subscription (Source: Apple - Developer News, `https://developer.apple.com/news/?id=12112017b`, accessed on 12/23/2025). Before that, Apple did not accommodate any form of TLF on the App Store. Thus, TLF is also now a mainstream option across all mobile app ecosystems for subscription-based apps (Google Play allowed native TLF since 2012). On the other hand, for software apps on desktop/laptop systems, free trials have been offered for a long time for one-time purchase products. At the same time, as of December 2025, free trials for *one-time purchase* mobile apps (through which full-fledged functionality or content is available for a limited time) are still not directly supported by either Apple App Store or Google Play Store. However, developers are allowed flexibility in hardcoding such features in the apps, should they choose to (developers can get creative with free apps with in-app purchases, somewhat approximating a native free-trial implementation). We clarify in Section 3.1, when introducing TLF, that in the context of the baseline model with 2 periods, it does not matter whether after the free trial we have perpetual or subscription-based license since only one period of use is left in the game anyways. Nevertheless, in Extension 3, in the context of a longer horizon with 3 periods (in Section 6.3 followed by details in E-companion E), we introduce *TLF-SUB* as different from TLF (to differentiate between subscription and perpetual licenses after the free trial). In sum, the inclusion of *TLF-SUB* model is highly relevant to the exploration of optimal go-to-market strategies for contemporary mobile app markets.

Supplementary References

Akçay, Y., T. Boyacı, and D. Zhang (2013). Selling with money-back guarantees: The impact on prices, quantities, and retail profitability. *Prod. & Oper. Management 22*(4), 777–791.

Autodesk (2025). Refund policies. `https://www.autodesk.com/benefits/refund-policy`, Accessed on 12/23/2025.

Bower, A. B. and J. G. Maxham III (2012). Return shipping policies of online retailers: Normative assumptions and the long-term consequences of fee and free returns. *J. Marketing 76*(5), 110–124.

Che, Y.-K. (1996). Customer return policies for experience goods. *J. Industrial Econ. 44*(1), 17–24.

Chen, H., W. Duan, and W. Zhou (2017). The interplay between free sampling and word of mouth in the online software market. *Decision Support Syst. 95*, 82–90.

Cheng, H. K., S. Li, and Y. Liu (2015). Optimal software free trial strategy: Limited version, time-locked, or hybrid? *Prod. & Oper. Management 24*(3), 504–517.

Cheng, H. K. and Y. Liu (2012). Optimal software free trial strategy: The impact of network externalities and consumer uncertainty. *Inform. Syst. Res. 23*(2), 488–504.

Datta, H., B. Foubert, and H. J. Van Heerde (2015). The challenge of retaining customers acquired with free trials. *J. Marketing Res. 52*(2), 217–234.

Davis, S., E. Gerstner, and M. Hagerty (1995). Money back guarantees in retailing: Matching products to consumer tastes. *J. Retailing 71*(1), 7–22.

Dey, D., A. Lahiri, and D. Liu (2013). Consumer learning and time-locked trials of software products. *J. Management Inform. Syst. 30*(2), 239–268.

Foubert, B. and E. Gijsbrechts (2016). Try it, you'll like it—or will you? The perils of early free-trial promotions for high-tech service adoption. *Marketing Sci. 35*(5), 810–826.

Goering, P. A. (1985). Effects of product trial on consumer expectations, demand, and prices. *J. Consumer Res. 12*(1), 74–82.

Hamm, B. C., M. Perry, and H. F. Wynn (1969). Effect of a free sample on image and attitude. *J. Advertising Res. 9*(4), 35–37.

He, Y., Q. Xu, and P. Wu (2020). Omnichannel retail operations with refurbished consumer returns. *Int. J. Prod. Res. 58*(1), 271–290.

Heiman, A., B. McWilliams, and D. Zilberman (2001). Demonstrations and money-back guarantees: Market mechanisms to reduce uncertainty. *J. Business Res. 54*(1), 71–84.

Heiman, A. and E. Muller (1996). Using demonstration to increase new product acceptance: Controlling demonstration time. *J. Marketing Res. 33*(4), 422–430.

Holmes, J. H. and J. D. Lett (1977). Product sampling and word of mouth. *J. Advertising Res. 17*(5), 35–40.

Jin, D., O. Caliskan-Demirag, F. Y. Chen, and M. Huang (2020). Omnichannel retailers' return policy strategies in the presence of competition. *Int. J. Prod. Econ. 225*, 107595.

Lee, Y.-J. and Y. Tan (2013). Effects of different types of free trials and ratings in sampling of consumer software: An empirical study. *J. Management Inform. Syst. 30*(3), 213–246.

Li, T. and X. Wang (2018). Offering free trials with consumer online reviews. In *Proc. The 2018 2nd Int. Conf. on Software and e-Business*, pp. 6–9.

Mehra, A. and R. L. Saha (2018). Utilizing public betas and free trials to launch a software product. *Prod. & Oper. Management 27*(11), 2025–2037.

Moorthy, S. and K. Srinivasan (1995). Signaling quality with a money-back guarantee: The role of transaction costs. *Marketing Sci. 14*(4), 442–466.

Nageswaran, L., S.-H. Cho, and A. Scheller-Wolf (2020). Consumer return policies in omnichannel operations. *Management Sci. 66*(12), 5558–5575.

Patel, P. C., C. Baldauf, S. Karlsson, and P. Oghazi (2021). The impact of free returns on online purchase behavior: Evidence from an intervention at an online retailer. *J. Oper. Management 67*(4), 511–555.

Reza, S., H. Ho, R. Ling, and H. Shi (2021). Experience effect in the impact of free trial promotions. *Management Sci. 67*(3), 1648–1669.

Schlereth, C., C. Barrot, B. Skiera, and C. Takac (2013). Optimal product-sampling strategies in social networks: How many and whom to target? *Int'l J. Electronic Commerce 18*(1), 45–72.

Sunada, T. (2018). Customer learning and revenue-maximizing trial design. Working paper. Department of Economics. Univ. Penn.

Suwelack, T., J. Hogreve, and W. D. Hoyer (2011). Understanding money-back guarantees: Cognitive, affective, and behavioral outcomes. *J. Retailing 87*(4), 462–478.

Wang, H., J. Lin, and L. Ran (2023). Time-locked free trial strategy in duopoly markets with switching costs. *4OR*, 1–43.

Wang, S. and G. F. Özkan-Seely (2018). Signaling product quality through a trial period. *Operations Res. 66*(2), 301–312.

Yoganarasimhan, H., E. Barzegary, and A. Pani (2023). Design and evaluation of optimal free trials. *Management Sci. 69*(6), 3220–3240.